Solutions for Fifty Challenging Problems in Probability

1

1. The Sock Drawer

A drawer contains red socks and black socks. When two socks are drawn at random, the probability that both are red is $\frac{1}{2}$ (a) How small can the number of socks in the drawer be? (b) How small if the number of black socks is even?

Solution for The Sock Drawer

Just to set the pattern, let us do a numerical example first. Suppose there were 5 red and 2 black socks; then the probability of the first sock's being red would be $5/(5 + 2)$. If the first were red, the probability of the second's being red would be $4/(4 + 2)$, because one red sock has already been removed. The product of these two numbers is the probability that both socks are red:

$$\frac{5}{5+2} \times \frac{4}{4+2} = \frac{5(4)}{7(6)} = \frac{10}{21}.$$

This result is close to $\frac{1}{2}$, but we need exactly $\frac{1}{2}$. Now let us go at the problem algebraically.

Let there be r red and b black socks. The probability of the first sock's being red is $r/(r + b)$; and if the first sock is red, the probability of the second's being red now that a red has been removed is $(r - 1)/(r + b - 1)$. Then we require the probability that both are red to be $\frac{1}{2}$, or

$$\frac{r}{r+b} \times \frac{r-1}{r+b-1} = \frac{1}{2}.$$

One could just start with $b = 1$ and try successive values of r, then go to $b = 2$ and try again, and so on. That would get the answers quickly. Or we could play along with a little more mathematics. Notice that

$$\frac{r}{r+b} > \frac{r-1}{r+b-1}, \qquad \text{for } b > 0.$$

Therefore we can create the inequalities

$$\left(\frac{r}{r+b}\right)^2 > \frac{1}{2} > \left(\frac{r-1}{r+b-1}\right)^2.$$

Taking square roots, we have, for $r > 1$,

$$\frac{r}{r+b} > \frac{1}{\sqrt{2}} > \frac{r-1}{r+b-1}.$$

From the first inequality we get

$$r > \frac{1}{\sqrt{2}}(r+b)$$

or

$$r > \frac{1}{\sqrt{2}-1}b = (\sqrt{2}+1)b.$$

From the second we get

$$(\sqrt{2}+1)b > r-1$$

or all told

$$(\sqrt{2}+1)b+1 > r > (\sqrt{2}+1)b.$$

For $b = 1$, r must be greater than 2.414 and less than 3.414, and so the candidate is $r = 3$. For $r = 3$, $b = 1$, we get

$$P(\text{2 red socks}) = \tfrac{3}{4} \cdot \tfrac{2}{3} = \tfrac{1}{2}.$$

And so the smallest number of socks is 4.

Beyond this we investigate even values of b.

b	r is between	eligible r	P(2 red socks)
2	5.8, 4.8	5	$\frac{5(4)}{7(6)} \neq \frac{1}{2}$
4	10.7, 9.7	10	$\frac{10(9)}{14(13)} \neq \frac{1}{2}$
6	15.5, 14.5	15	$\frac{15(14)}{21(20)} = \frac{1}{2}$

And so 21 socks is the smallest number when b is even. If we were to go on and ask for further values of r and b so that the probability of two red socks is $\frac{1}{2}$, we would be wise to appreciate that this is a problem in the theory of numbers. It happens to lead to a famous result in Diophantine Analysis obtained from Pell's equation.* Try $r = 85$, $b = 35$.

*See for example, W. J. LeVeque, *Elementary theory of numbers*, Addison-Wesley, Reading, Mass., 1962, p. 111.