Reprinted from
The Mathematics Teacher
(1961), **54**, *pp. 411-412*

17. Optimal Length of Play for a Binomial Game

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In a game where A plays against B, how many trials should A choose to play to maximize his chance of success?

A game between players A and B consists of $N(=2n)$ independent trials. One point is awarded on each trial—to player A with probability $p < \frac{1}{2}$, or to player B with probability $q(=1-p)$. If player A wins more than half the points, he wins a prize. How shall he choose N to maximize his chance of success?[1] Player A knows the value of p.

At first blush, most people notice that the game is unfair, and therefore as N increases, the expected value of the difference (A's points $-$ B's points) grows more and more negative. They conclude that A should play as little as he can and still win—that is, two trials.

Had an odd number of trials been allowed, this reasoning based on expected values would have been correct, and A should play only one trial (and this note would not have been written). But with an even number of trials there are two effects at work, (1) the bias in favor of B, and, opposing that at first (2) the redistribution of the probability in the middle term of the binomial distribution (the probability of a tie) as the number of trials increases.

Consider, for a moment, a fair game ($p = \frac{1}{2}$). Then the larger N, the larger A's chance to win because as $2n$ increases, the probability of a tie tends to zero, and in the limit A's chance to win is $\frac{1}{2}$. For $N = 2, 4, 6$, his probabilities are $\frac{1}{4}$, $\frac{5}{16}$, $\frac{22}{64}$. Continuity suggests that for p slightly less than $\frac{1}{2}$, A should play a large but finite number of games. But if p is small, $N = 2$ should be optimal for A. It turns out that for $p < \frac{1}{3}$, $N = 2$ is optimal.

[1] P.G. Fox originally alluded to a result which gives rise to this game in "A Primer for Chumps," which appeared in the *Saturday Evening Post*, November 21, 1959, and discussed the idea further in private correspondence arising from that article in a note entitled "A Curiosity in the Binomial Expansion—and a Lesson in Logic." I am indebted to Clayton Rawson and John Scarne for alerting me to Fox's paper.

An approximation

Let us use the usual normal approximation to estimate the optimal value of N. Let the random variable X be A's number of points in $N = 2n$ trials. We wish to maximize $P(X \geq n+1)$. Let

$$z = \frac{(n+1) - 2np - \frac{1}{2}}{\sqrt{2npq}}$$

where the $\frac{1}{2}$ is the usual continuity correction.

Then if Z is a standard normal random variable (zero mean and unit variance), $P(Z > z)$ approximates $P(X > n)$. The smaller z, the larger $P(Z > z)$, so we wish to find the value of n that minimizes z for a fixed $p < \frac{1}{2}$.

Standard calculus methods yield for the value for N that minimizes z

$$N = 2n = \frac{1}{1-2p}.$$

For example, if $p = 0.49$, $N = 50$, and, wonder of wonders, this value of N is not just approximate, but is exactly the one that maximizes $P(X \geq n+1)$. Can such good fortune continue? Consider $p = \frac{1}{3}$, then the estimated value of N is 3. It turns out that the surrounding even values of N, 2 and 4, give identical probabilities of a win for A, namely $\frac{1}{9}$; and this $\frac{1}{9}$ is the optimal probability that A can achieve!

These results, and others like them suggest that the nearest even integer to $1/(1-2p)$ is the optimal value of N, unless $1/(1-2p)$ is an odd integer and that then both neighboring integers are optimal. To assist in the proof of this conjecture, we let P_{2n} be the probability that $X \geq n+1$ in a game of $2n$ trials:

$$P_{2n} = \sum_{x=n+1}^{2n} \binom{2n}{x} p^x q^{2n-x}.$$

In a game of $2n+2$ trials $P(X \geq n+2)$ is

$$P_{2n+2} = \sum_{x=n+2}^{2n+2} \binom{2n+2}{x} p^x q^{2n+2-x}.$$

A game composed of $2n+2$ trials can be regarded as having been created by adding two trials to a game of $2n$ trials. Unless player A has won either n or $n+1$ times in the $2n$ game, his status as a winner or loser cannot differ in the $2n+2$ game from that in the $2n$ game.

Except for these two possibilities, P_{2n+2} would be identical with P_{2n}. These exceptions are (1) that having $n+1$ successes in the first $2n$ trials, A loses the next two, thus reducing his probability of winning in the $2n+2$ game by

$$q^2 \binom{2n}{n+1} p^{n+1} q^{n-1};$$

or (2) that having won n trials in the $2n$ game he wins the next two, increasing his probability by

$$p^2 \binom{2n}{n} p^n q^n.$$

If $N = 2n$ is the optimal value, then both $P_{N-2} \leq P_N$ and $P_N \geq P_{N+2}$ must hold. The results of the previous paragraph imply that these inequalities are equivalent to

$$\begin{aligned} q^2 \tbinom{2n-2}{n} p^n q^{n-2} &\leq p^2 \tbinom{2n-2}{n-1} p^{n-1} q^{n-1}; \\ q^2 \tbinom{2n}{n+1} p^{n+1} q^{n-1} &\geq p^2 \tbinom{2n}{n} p^n q^n \end{aligned} \tag{1}$$

or, after some simplifications (we exclude the trivial case $p = 0$),

$$(n-1)q \leq np; \quad nq \geq (n+1)p. \tag{2}$$

These inequalities yield, after a little algebra, the condition

$$\frac{1}{1-2p} - 1 \leq 2n \leq \frac{1}{1-2p} + 1 \tag{3}$$

Thus unless $1/(1-2p)$ is an odd integer, N is uniquely determined as the nearest even integer to $1/(1-2p)$. When $1/(1-2p)$ is an odd integer, both adjacent even integers give the same optimal probability. And we can incidentally prove that when $1/(1-2p) = 2n+1$, $P_{2n} = P_{2n+2}$.

Generalization

I.R. Savage suggested to the author that the game be generalized. Again with the probability $p(< \frac{1}{2})$ of winning a single point, suppose that to win the game of N trials, one must have r more points than one's opponent. Again we want to find the value of N (unrestricted) that maximizes the probability of winning. The normal approximation suggests that the optimum value of $N \approx (r-1)/(1-2p)$. Suitably interpreted, this approximation is satisfactory as an exact solution. The methods used above in the exact solution for $r = 2$ and N an even integer can be extended. It turns out after some algebra that for odd values of r, the optimum value of N is the positive odd integer nearest $(r-1)/(1-2p)$, while for r an even integer, the optimum value of N is the positive even integer nearest $(r-1)/(1-2p)$.

The distinction between odd and even values of r can matter substantially. For example, let $r = 3$, $p = 0.4$, then direct application of the approximate formula gives $N = 10$, with probability 0.05476 of winning by at least 3 points, which for an even value of N means by at least 4 points. On the other hand for $N = 9$ or 11, the probability of winning by at least 3 points (a win by exactly 3 points is now achievable) is 0.09935. Furthermore, the value $N = 10$ is not the best *even* value of N to choose, because then winning by 3 implies winning by 4, and the optimum N for $r = 4$ is $N = 14$ (or 16), with probability of winning 0.05832.