RECREATIONAL MATHEMATICS COLLOQUIUM II

April 27th - April 30th 2011

University of Évora

Conference Proceedings

Jorge Nuno Silva (Ed.)

http://ludicum.org/rm11/

Proceedings of the

Recreational Mathematics Colloquium II

Jorge Nuno Silva (Ed.)

Associação Ludus

Lisboa, 2012

ISBN: 978-989-97346-2-3

Editor: Associação Ludus

R. da Escola Politécnica, 56

1250-102 Lisboa - PORTUGAL

Email: ludus@ludicum.org

URL: http://ludicum.org

Typesetting: Joaquim Rosa
(jrosa@lasige.di.fc.ul.pt)

Table of Contents

Introduction

A major event in Recreational Mathematics held every other year in Europe! One tradition we are proud of having initiated. This was only the second in the series of colloquia, but the enthusiasm is overwhelming and we can foresee a bright future to this initiative.

For a few days Évora was the epicenter of mathematical fun. Those who experienced it will never forget the historic walls of the town, its monuments and narrow alleys, the conferences, the conversations during the breaks, the stimulating pleasure of sharing beautiful mathematical ideas.

In these proceedings you can read about invisible objects, the songs of the birds, juggling, games, cartoons, 13$^{\text{th}}$ century recreations, quilts, literature, ...

These texts will give an idea to their readers of the mathematical celebration that we all lived in Évora, and, at the same time, will point the way to *Recreational Mathematics Colloquium III*, to be held at the University of Azores, Ponta Delgada, in April 3-6, 2013.

Murtal, December 2011

Jorge Nuno Silva

A Very Useful Pythagorean Tree

Alda Carvalho
ISEL and CEMAPRE/ISEG
ADM, ISEL, Rua Conselheiro Emídio Navarro, 1
1959-007, Lisboa, Portugal
acarvalho@adm.isel.pt

Carlos Pereira dos Santos
ISEC and CIMA/UE
Alameda das Linhas de Torres, 179
1750-142, Lisboa, Portugal
carlos.santos@isec.universitas.pt

Abstract

In this work, we resume some results about a particular Pythagorean tree, described in some recent works (Bernhart & Lee Price, 2007; Lee Price, 2008). We illustrate how H. Lee Price used his work as a tool to perform quick mental calculations. This Pythagorean tree, based in Fibonacci boxes, is also strongly related to the two most famous mathematical artifacts: the *Plimpton 322* and the *Rhind Papyrus*.

Introduction

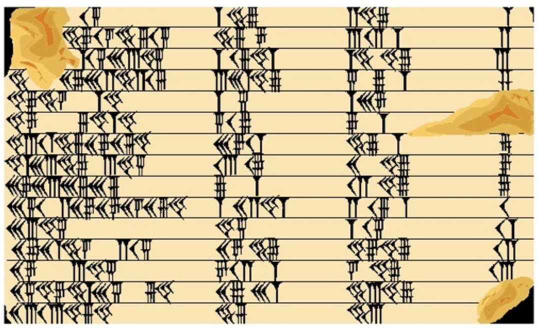
A primitive Pythagorean triple is a Pythagorean triple (a, b, c) such that $GCD(a, b, c) = 1$. Of course, in a Pythagorean triple, a, b, and c are positive integers and $a^2 + b^2 = c^2$. The early Greeks gave the following closed expression for the primitive Pythagorean triples:

$$(p^2 - l^2, 2pl, p^2 + l^2), p > l, GCD(p^2 - l^2, 2pl) = 1$$

An amazing artefact directly related to this subject is the *Plimpton 322*.

Figure 1: Plimpton 322 (1800 BC), Babylonian clay tablet, 322th in the G.A. Plimpton Collection at Columbia University.

In this tablet, the second and the third columns correspond to a leg and the hypotenuse of a right triangle and the first column corresponds to the ratio between the legs. The numbers are expressed in sexagesimal notation, used by the Babylonians.



l_1^2/l_2^2	l_1	h
59: :15: : : : :	:1:59	:2:49
56:56:58:14:50:6:15:	:56:7	1:20:25
55:7:41:15:33:45: :	1:16:41	1:50:49
53:10:29:32:52:16: :	3:31:49	5:9:1
48:54:1:40: : : :	:1:5	:1:37
47:6:41:40: : : :	:5:19	:8:1
43:11:56:28:26:40: :	:38:11	:59:1
41:33:45:14:3:45: :	:13:19	:20:49
38:33:36:36: : : :	:8:1	:12:49
35:10:2:28:27:24:26:40	1:22:41	2:16:1
33:45: : : : : :	45	1:25
29:21:54:2:15: : :	:27:59	:48:49
27: :3:45: : : :	:2:41	:4:49
25:48:51:35:6:40: :	:29:31	:53:49
23:13:46:40: : : :	: :28	: :53

l_1^2/l_2^2	l_1	h	Line	l_2
0,98340278	119	169	1	120
0,94915855	3367	4825	2	3456
0,91880213	4601	6649	3	4800
0,88624791	12709	18541	4	13500
0,81500772	65	97	5	72
0,7851929	319	481	6	360
0,71998368	2291	3541	7	2700
0,69270942	799	1249	8	960
0,64266944	481	769	9	600
0,58612257	4961	8161	10	6480
0,5625	0,75	1,25	11	1
0,48941684	1679	2929	12	2400
0,45001736	161	289	13	240
0,43023882	1771	3229	14	2700
0,38716049	28	53	15	45

Figure 2: Right Triangles in Plimpton 322.

Another important artefact indirectly related to this subject (we will see why) is the *Rhind Papyrus*.

Figure 3: Rhind Papyrus(1650 BC), Egiptian Papyrus, British Museum.

In this papyrus, among other things, we can find a $\frac{2}{n}$-table. The fractions $\frac{2}{n}$ for odd n ranging from 3 to 101 are expressed as sums of unit fractions. For example, $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$.

Following, we will describe a well known Pythagorean tree related two these two artifacts (probably the two most famous mathematical artifacts of Antiquity). Consider one of the most beautiful Pythagorean trees, the *Vogeler's diagram*, described in *The Book of Numbers* (Conway & Guy, 1996):

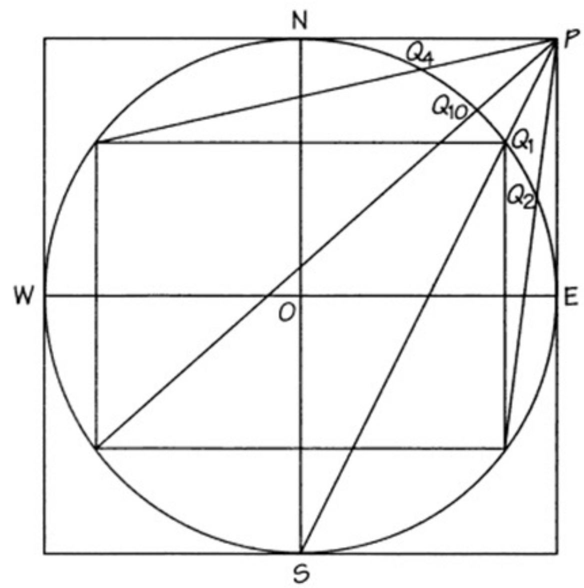


Figure 4: *The Book of Numbers*, page 172

John H. Conway and Richard K. Guy explained in their book:

Roger Vogeler has shown that the process sketched in Figure 4 gives each Pythagorean fraction $\frac{x}{y}$ just once. A circle is inscribed in a square. Join a corner of the square, P, to where it touches the circle at S or W. The other place, Q_1, where this line cuts the circle is one corner of a $(3,4,5)$-shaped rectangle. If we join the other corners to P, we find the points Q_2, Q_4, and Q_{10}, on the circle which are the corners of $(5,12,13)$-, $(8,15,17)$- and $(20,21,29)$-shaped rectangles. If you join P to the corners of each new rectangle, you will discover further rectangles, and so on forever.

So, each corner is the "father" of 3 new Pythagorean triples. Next figure shows the three first iterations.

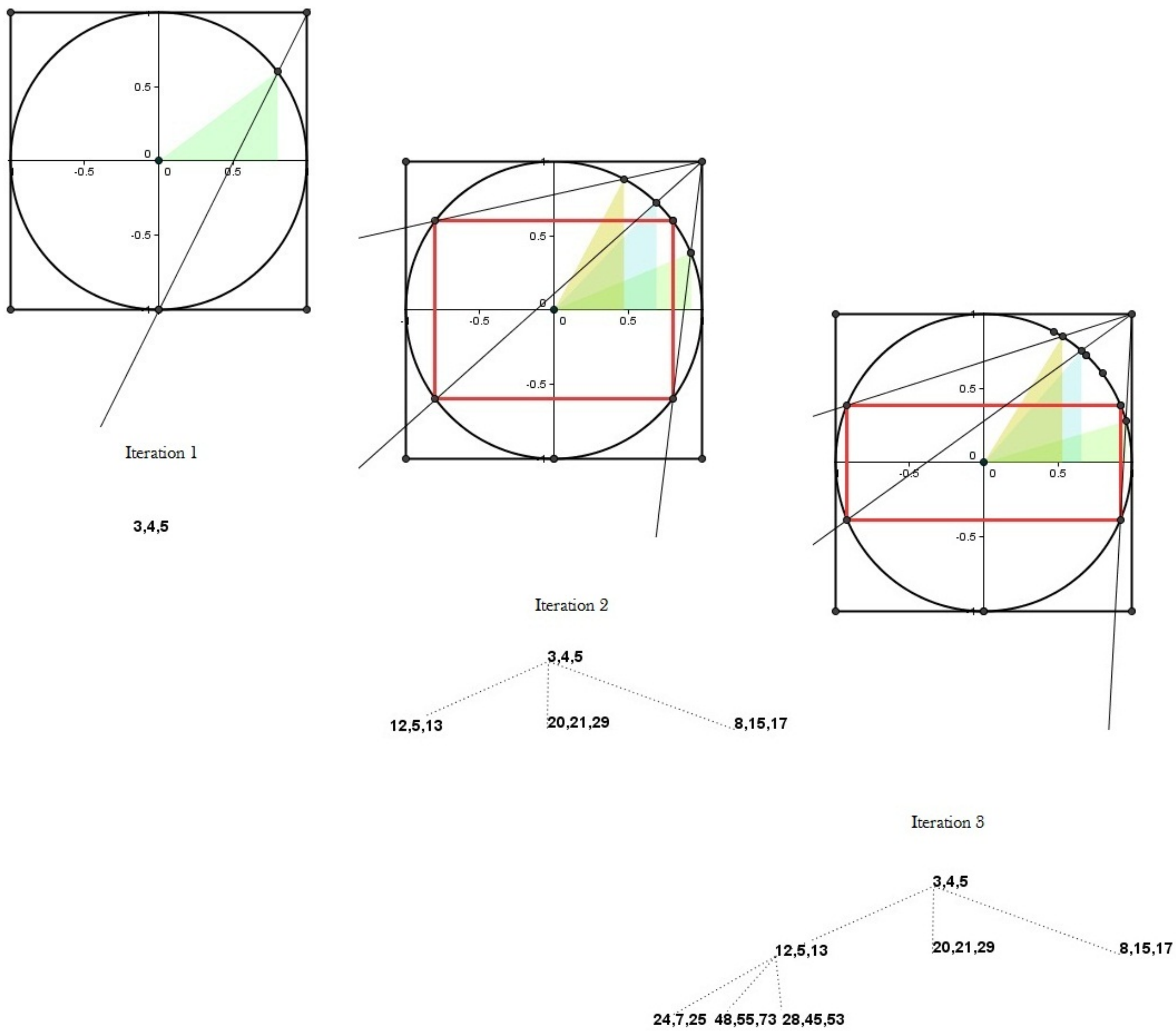


Figure 5: Vogeler's Diagram

There are several recursive processes generating trees of this kind. In 1963, F.J.M. Barning described an infinite, planar, ternary tree whose nodes are the set of primitive Pythagorean triples (Barning, 1963). Some years later, A. Hall, independently, discovered the same tree (Hall, 1970). In the next sections we will describe some work done by H. Lee Price using the Fibonacci rule for the construction of a tree of this kind (Bernhart & Lee Price, 2007; Lee Price, 2008). It is really an interesting approach with clear proofs, providing very useful tips for mental calculation.

Primitive Pythagorean Triples: Some Theorems

Every triangle is related to a set of four circles:

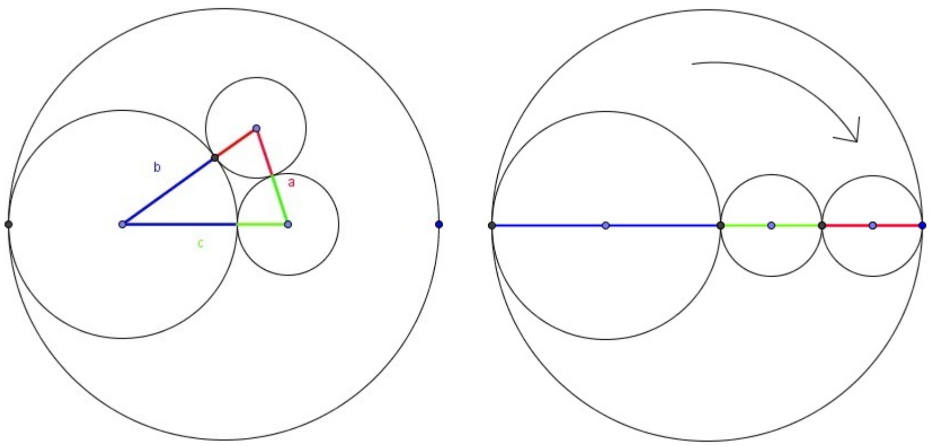


Figure 6: Four Circles.

Being r_4 the radius of the largest circle and r_1, r_2, and r_3 the radii of the smaller circles, visual inspection allows the following conclusions:

- $a = r_1 + r_2 = r_4 - r_3$
- $b = r_1 + r_3 = r_4 - r_2$
- $c = r_2 + r_3 = r_4 - r_1$

and

- $r_1 = \frac{1}{2}(a + b - c)$
- $r_2 = \frac{1}{2}(a - b + c)$
- $r_3 = \frac{1}{2}(-a + b + c)$
- $r_4 = \frac{1}{2}(a + b + c) = r_1 + r_2 + r_3$

For right triangles, we have

- $r_2.r_3 = \frac{1}{4}(a - b + c)(-a + b + c) = \frac{1}{4}(-a^2 + ab + ac + ab - b^2 - bc - ac + bc + c^2) = \frac{1}{2}ab$
- $r_1.r_4 = \frac{1}{4}(a + b - c)(a + b + c) = \frac{1}{4}(a^2 + ab + ac + ab + b^2 + bc - ac - bc - c^2) = \frac{1}{2}ab$

So, for right triangles, we have the fundamental identity:

$$\boxed{r_2.r_3 = r_1.r_4}$$

Now, we list some important theorems underlying the third section.

Theorem 1. *For primitive Pythagorean triples, there are exactly two even r_i's.*

Proof. From $r_2r_3 = r_1r_4$ and $r_4 = r_1 + r_2 + r_3$ we have

$$\begin{aligned}
& r_2r_3 = r_1r_4 \\
\Leftrightarrow\quad & r_2r_3 = r_1(r_1 + r_2 + r_3) \\
\Leftrightarrow\quad & r_2r_3 = r_1^2 + r_1r_2 + r_1r_3 \\
\Leftrightarrow\quad & r_1^2 = r_2r_3 - r_1r_2 - r_1r_3 \\
\Leftrightarrow\quad & r_1^2 = r_3(r_2 - r_1) - r_1r_2 \\
\Leftrightarrow\quad & 2r_1^2 = r_3(r_2 - r_1) - r_1r_2 + r_1^2 \\
\Leftrightarrow\quad & 2r_1^2 = r_3(r_2 - r_1) - r_1(r_2 - r_1) \\
\Leftrightarrow\quad & 2r_1^2 = (r_2 - r_1)(r_3 - r_1)
\end{aligned}$$

1. Suppose that r_i are all odd. Then, $r_2 - r_1$ and $r_3 - r_1$ are even and r_1^2 is odd. Since $2 \times \text{odd} = \text{even} \times \text{even}$ is impossible, one of the r_i must be even.

2. Suppose that exactly one of the r_i's is even. Then, $r_2.r_3 = r_1.r_4$ is impossible.

3. Suppose that exactly three of the r_i's are even. Then, $r_2.r_3 = r_1.r_4$ is impossible.

4. Suppose that the r_i are all even. Then, the Pythagorean triple is not primitive because all the r_i's are divisible by 2.

Therefore, there must be exactly two even r_i's. □

Important Remark: From $r_2r_3 = r_1r_4$, we conclude that if r_1 is even, then r_4 is odd and if r_4 is even, then r_1 is odd. The same for r_2 and r_3. So, **it is possible** to consider an order for r_1, r_2, r_3, and r_4 in such a way that the parities alternate (changing r_2 with r_3 if necessary). From now on, we will consider that order.

From the fundamental identity $r_2r_3 = r_1r_4$, we have

$$\frac{r_1}{r_2} = \frac{r_3}{r_4} = \frac{a+b-c}{a+c-b} = \frac{(a+b-c)(a+b+c)}{(a+c-b)(a+c+b)} = \frac{a^2+2ab+b^2-c^2}{a^2+2ac+c^2-b^2} = \frac{2ab}{2a^2+2ac} = \frac{b}{a+c}$$

And, because $\frac{b}{a+c} = \frac{b(a-c)}{(a+c)(a-c)} = \frac{b(a-c)}{-b^2} = \frac{c-a}{b}$, we have the following equalities (the second line has an analogous argument):

$$\frac{r_1}{r_2} = \frac{r_3}{r_4} = \frac{b}{a+c} = \frac{c-a}{b}$$

$$\frac{r_1}{r_3} = \frac{r_2}{r_4} = \frac{a}{b+c} = \frac{c-b}{a}$$

Theorem 2. *Consider* $\frac{r_1}{r_2} = \frac{r_3}{r_4} = \frac{q}{p}$ *and* $\frac{r_2}{r_4} = \frac{r_1}{r_3} = \frac{q'}{p'}$ *where* $\frac{q}{p}$ *and* $\frac{q'}{p'}$ *are reduced fractions. Then,*

$$r_1 = qq',\ r_2 = q'p,\ r_3 = p'q,\ r_4 = p'p.$$

Proof. Because $\frac{r_1}{r_2} = \frac{r_3}{r_4} = \frac{q}{p}$ and $\frac{q}{p}$ is reduced, we have integers x and y such that

$$\begin{array}{ll} r_1 = xq & r_2 = xp \\ r_3 = yq & r_4 = yp \qquad (*) \end{array}$$

So, we must have $GCD(x, y) = 1$. If not, r_1, r_2, r_3 and r_4 would share a common divisor contradicting the primitivity of the Pythagorean triple.

Finally, replacing $r_1 = xq$, $r_2 = xp$, $r_3 = yq$, and $r_4 = yp$, we have

$$\frac{r_2}{r_4} = \frac{r_1}{r_3} = \frac{x}{y} = \frac{q'}{p'}.$$

Because $\frac{x}{y}$ and $\frac{q'}{p'}$ are both reduced, we have $x = q'$ and $y = p'$. Replacing in (*), we obtain

$$\begin{array}{ll} r_1 = q'q & r_2 = q'p \\ r_3 = p'q & r_4 = p'p \end{array}$$

□

Theorem 3. *Consider* p, q, p', q' *of the Theorem 2. The only even number is* p *or* q*.*

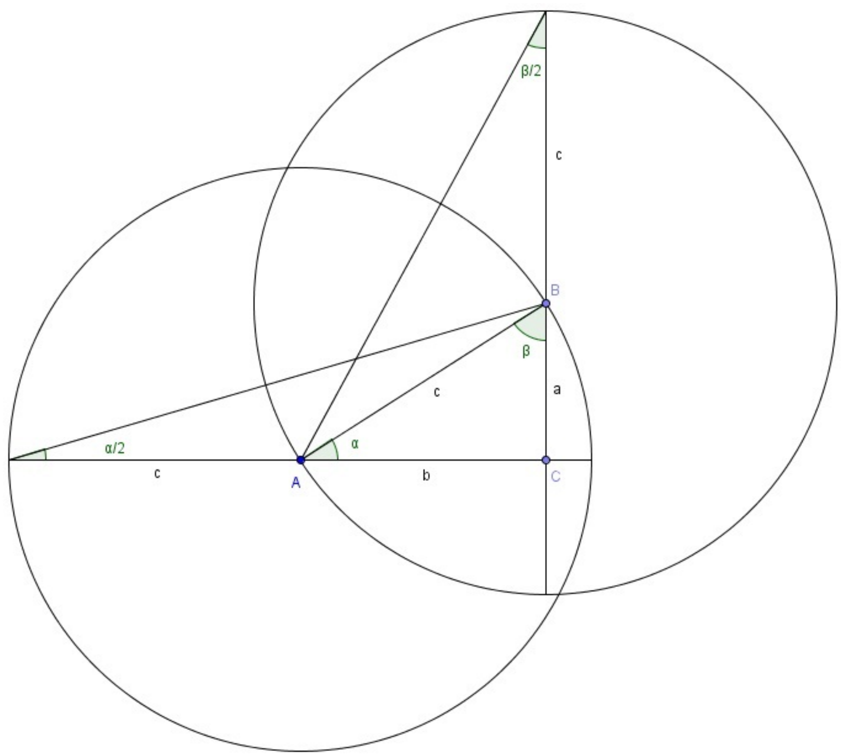
Proof. We considered an order for r_1, r_2, r_3, and r_4 in such a way that the parities alternate. So, when we observe the equalities $\frac{r_1}{r_3} = \frac{r_2}{r_4} = \frac{q'}{p'}$, one of the two first fractions has odd numerator and odd denominator. Therefore, q' and p' are odd numbers.

From Theorems 1 and 2, we conclude that exactly one of the p and q is even. □

Theorem 4. *Consider p, q, p', q' of the Theorem 2. Then,*

$$\frac{q}{p} + \frac{q'}{p'} + \frac{qq'}{pp'} = 1$$

Proof. Consider a right triangle ABC and the following picture:



We have $\alpha + \beta = \frac{\pi}{2}$ and $\frac{\alpha}{2} + \frac{\beta}{2} = \frac{\pi}{4}$. So,

$$\begin{aligned}
& \tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = 1 \\
\Leftrightarrow \quad & \frac{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)}{1 - \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)} = 1 \\
\Leftrightarrow \quad & \frac{\frac{a}{c+b} + \frac{b}{c+a}}{1 - \frac{a}{c+b}\frac{b}{c+a}} = 1 \\
\Leftrightarrow \quad & \frac{\frac{q'}{p'} + \frac{q}{p}}{1 - \frac{q'}{p'}\frac{q}{p}} = 1 \\
\Leftrightarrow \quad & \frac{q'}{p'} + \frac{q}{p} = 1 - \frac{q'}{p'}\frac{q}{p} \\
\Leftrightarrow \quad & \frac{q'}{p'} + \frac{q}{p} + \frac{q'}{p'}\frac{q}{p} = 1
\end{aligned}$$

□

Theorem 5. *Consider p, q, p', q' of the Theorem 2. Then,*

$$p - q = q' \quad p + q = p' \quad \frac{1}{2}(p' - q') = q \quad \frac{1}{2}(p' + q') = p$$

Proof. From Theorem 4,

$$\begin{aligned} & \frac{q}{p} + \frac{q'}{p'} + \frac{qq'}{pp'} = 1 \\ \Leftrightarrow \quad & \frac{qp'}{pp'} + \frac{q'p}{pp'} + \frac{qq'}{pp'} = \frac{pp'}{pp'} \\ \Leftrightarrow \quad & qp' + qq' = pp' - pq' \\ \Leftrightarrow \quad & \frac{q}{p} = \frac{p' - q'}{p' + q'} \end{aligned}$$

Analogous argument for $\frac{q'}{p'} = \frac{p-q}{p+q}$.

By Theorem 3, p' and q' are odd and, so, $p' - q'$ and $p' + q'$ are even. Therefore $\frac{p'-q'}{2}$ and $\frac{p'+q'}{2}$ are integers. Because $\frac{q}{p}$ is reduced and $\frac{q}{p} = \frac{\frac{p'-q'}{2}}{\frac{p'+q'}{2}}$, there is an integer z such that

$$zq = \frac{p' - q'}{2} \;\; (*1) \qquad zp = \frac{p' + q'}{2} \;\; (*2).$$

Adding and subtracting the two equations, $z(p+q) = p'$ $(*3)$ and $z(p-q) = q'$ $(*4)$. Finally, because $\frac{q'}{p'} = \frac{p-q}{p+q}$ and $\frac{q'}{p'}$ is reduced, $z = 1$.

Replacing $z = 1$ in $(*1)$, $(*2)$, $(*3)$, and $(*4)$, we finish the proof.

□

Theorem 6. *Consider p, q, p', q' of the Theorem 2. Then,*

$$a = q'p' \quad b = 2qp \quad c = qp' + q'p$$

Proof. We know that $a = r_1 + r_2$, $b = r_1 + r_3$ and $c = r_2 + r_3$. By Theorem 2,

- $a = qq' + q'p = q'(q + p) =_{(\text{Th5})} q'p'$
- $b = qq' + p'q = q(q' + p') =_{(\text{Th5})} 2qp$
- $c = q'p + p'q$

□

We finish this list with a theorem establishing the relation to the Rhind Papyrus:

Theorem 7. *Consider p, q, p', q' of the Theorem 2. Then,*

$$\frac{2}{q'p'} = \frac{1}{q'p} + \frac{1}{pp'}$$

Proof. We start arguing that $\frac{2}{r_1} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}$.

$$\begin{aligned}
& \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \\
= & \frac{r_2r_3r_4 + r_1r_3r_4 + r_1r_2r_4 + r_1r_2r_3}{r_1r_2r_3r_4} \\
= & \frac{r_2r_3r_4 + r_1r_3r_4 + r_1r_2r_4 + r_1r_2r_3}{(r_2r_3)^2} \\
= & \frac{r_1 + r_2 + r_3 + r_4}{r_2r_3} \\
= & \frac{2r_4}{r_1r_4} \\
= & \frac{2}{r_1}
\end{aligned}$$

The last equation allows the conclusion:

$$\begin{aligned}
& \frac{2}{r_1} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \\
\Leftrightarrow\ & \frac{1}{r_1} = \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \\
\Leftrightarrow\ & \frac{1}{qq'} = \frac{1}{q'p} + \frac{1}{qp'} + \frac{1}{pp'} \\
\Leftrightarrow\ & \frac{1}{qq'} - \frac{1}{qp'} = \frac{1}{q'p} + \frac{1}{pp'} \\
\Leftrightarrow\ & \frac{p' - q}{p'qq'} = \frac{1}{q'p} + \frac{1}{pp'} \\
\Leftrightarrow\ & \frac{2q}{qq'p'} = \frac{1}{q'p} + \frac{1}{pp'} \\
\Leftrightarrow\ & \frac{2}{q'p'} = \frac{1}{q'p} + \frac{1}{pp'}
\end{aligned}$$

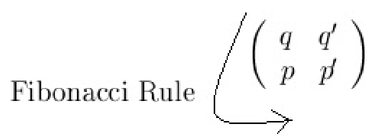
□

A Pythagorean Tree Constructed with Fibonacci Boxes: Mental Calculations

The theorems exposed in the second section allow us very quick calculations related to primitive Pythagorean triples. Consider the following *Fibonacci box*:

$$\begin{pmatrix} q & q' \\ p & p' \end{pmatrix}$$

By Theorem 5, $p = q' + q$ and $p' = q + p$. So, if we start with an odd q' and a q such that $GCD(q', q) = 1$, we just have to apply the Fibonacci rule to complete the box:



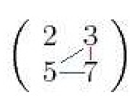
More, by Theorem 6, $a = q'p'$ is the second column product, $b = 2qp$ is the double of the first column product, and $c = qp' + q'p$ is the permanent (determinant of the matrix but adding instead of subtracting).

Let us look at a quick example. A teacher is writing on the blackboard and he needs a Pythagorean triple to "build" a good exercise. He chooses $q' = 3$ and $q = 2$ to start. After, he applies the Fibonacci rule getting $p = 3 + 2 = 5$ and $p' = 2 + 5 = 7$. The Fibonacci box and the primitive Pythagorean triple are

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$$

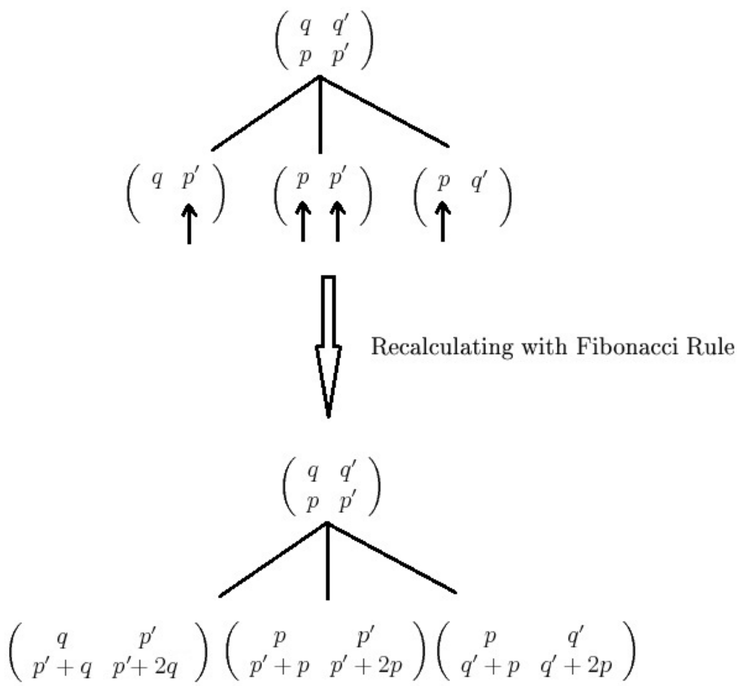
$$a = 3 \times 7 = 21,\ b = 2 \times 2 \times 5 = 20,\ \textit{and}\ c = 2 \times 7 + 3 \times 5 = 29$$

More, the Theorem 7 allows a quick calculation to obtain a "Rhind's decomposition":



$$\frac{2}{3 \times 7} = \frac{1}{3 \times 5} + \frac{1}{5 \times 7} \Leftrightarrow \frac{2}{21} = \frac{1}{15} + \frac{1}{35}$$

We can get the Barning-Hall tree introducing the following "promotion rule":



Starting with $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, we get an infinite, planar, ternary tree:

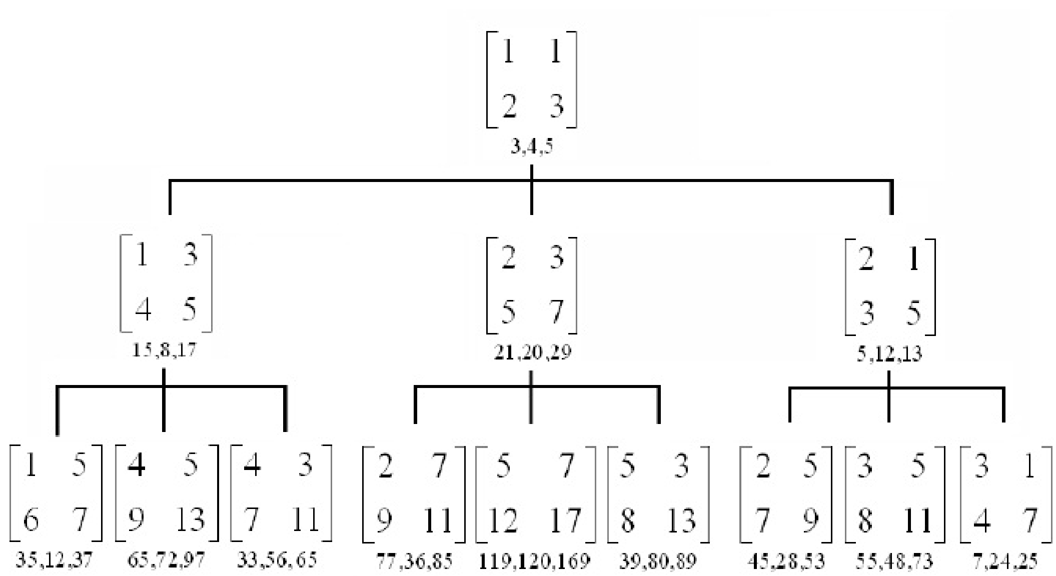


Figure 7: The Barning-Hall tree (first 3 generations).

In fact, it is easy to argue that every Fibonacci box has exactly one position in this tree (Bernhart & Lee Price, 2007). To justify this, we can see that every Fibonacci box is "demoted" in exactly one way. Consider a Fibonacci box

$$\begin{pmatrix} y & x \\ x+y & x+2y \end{pmatrix}$$

There are three "candidates" to be the its ascendent:

$$\begin{pmatrix} y & x-2y \\ x-y & x \end{pmatrix} \begin{pmatrix} x-y & 2y-x \\ y & x \end{pmatrix} \begin{pmatrix} y-x & x \\ y & 2y-x \end{pmatrix}$$

If $x < y$, the third candidate is the only possibility. If $x > y$, the only possibility is the first or the second candidate depending on whether $2y - x$ is positive or not. Of course, if $x = y$, we have the initial node $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

References

Barning, F. J. M. (1963). "On Pythagorean and quasi-Pythagorean triangles and a generation process with the help of unimodular matrices", (Dutch) Math. Centrum Amsterdam Afd. Zuivere Wisk. ZW-001.

Bernhart, F. & Lee Price, H. (2007). "Heron's Formula, Descartes Circles, and Pythagorean Triangles", arXiv:math.MG/0701624.

Conway, J. & Guy, R. K. (1996). *The Book of Numbers*, Springer.

Hall, A. (1970). "Geneology of Pythagorean Triads", Math. Gazette 54, 390, 377-379.

Lee Price, H. (2008). "The Pythagorean Tree: A New Species", arXiv.org:0809.4324.

Treason Game

João Cabral
Departamento de Matemática
Universidade dos Açores, Portugal
jcabral@uac.pt

Helena Melo
Departamento de Matemática
Universidade dos Açores, Portugal
hmelo@uac.pt

Abstract

Treason is a two player game played with the usual rules of the checkers game, but played in a different board, with the needed adaptations. Instead of squares we have triangles, and the initial position of the pieces is similar to the checkers game. The board has also in the main diagonal a free battle zone that allows the players to diversify the strategies until a level of complexity very high.

One of the main differences between this game and the checkers is that a player has, at least, from one triangle, three possibility lines to play.

The name of the game comes from the most important move that allows one player to promote pieces, common people, to Spies, behind enemy lines and to convert enemies to their side. This move gives a chance to Spies to sweep literally the enemy pieces by the back.

Setup Your Treason board

Treason is played on a board made up of triangles, very similar to the usual and very well known checkers game, but much richer in strategies.

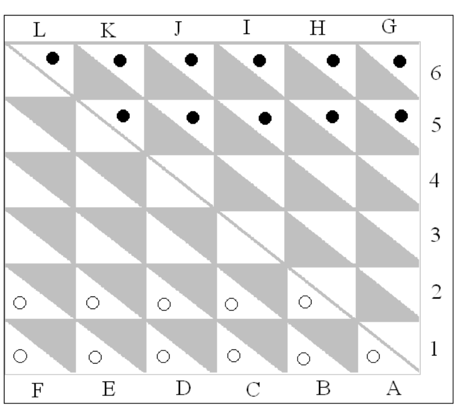


Figure 1: Treason board

Treason is a game for two players. Each player receives eleven, flat disk-like pieces which are placed on the white triangles in the manner indicated in the diagram at the left, figure 1. The darker coloured pieces are designated black, and the lighter colour is designated white. White always moves first. There is some advantage to moving first, but on the beginner level it is very slight.

While playing, you may record your game. To record a move, simply write down the triangle, the move is from and then the number of the triangle where the piece was moved. This is convenient for discussing games and strategies.

The square board is divided in two triangles, the North, where we put the black pieces, and the South, where we put the white pieces. The system of identification in the North is made by columns G, H, ..., L and lines 6, 5,...,1. In the South the triangles are identified by columns A, B, ..., F and lines 6, 5, ...1. This system of identification assures us an unique representation of each white triangle. For example, we have G1 and A1 as identification for the triangles that are adjacent in the lower right corner of the board.

The line L6, K6, J6, I6, H6, G6 is called the promotion line for the white pieces. Each white piece that achieves to reach there is promoted to a higher level of movement and the piece is called White Spy.

The line A1, B1, C1, D1, E1, F1 is called the promotion line for the black pieces. Each black piece that achieves to reach there is promoted to a higher level of movement and the piece is called Black Spy.

General Rules for Treason

Now that you have set up the board, you are ready to begin play. First, determine who is to be "black". You can use any method for this you wish, flip a coin, alternate, etc. However, the most common method, in amateur play for board games, is for one of the players to take one colour piece in each hand and hold out his hands before him. The other player chooses a hand; the colour piece in that hand determines the colour with which he plays.

The objective is to eliminate all opposing pieces or to create a situation in which it is impossible for your opponent to make any move. Normally, the victory will be due to complete elimination.

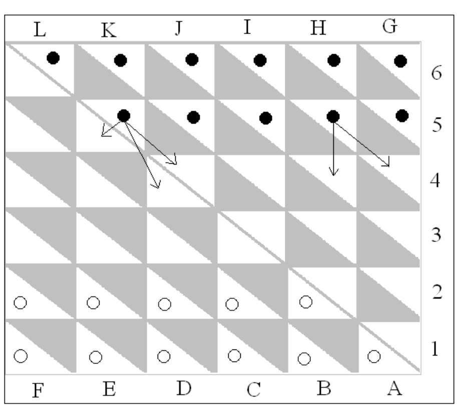


Figure 2: Allowed movements

White moves first and play proceeds alternately. The pieces may only move forward, to the left or to the right, forward left or forward right. There are two types of moves that can be made, *capturing moves* and *non-capturing moves*.

Non-capturing moves are simply moves from one triangle to another with common vertices or common sides. (Note that the black triangles are never used.) For example from position K5, the piece can go to E5 or D4 or J4; from position H5 the piece can go to H4 and G4 (see figure 2). The pieces cannot go to a triangle already occupied with another piece, or jump over other pieces. The following tables show the allowed movements for all positions. The allowed move is represented by "1" and the not allowed is represented by "0". Exemplifying we can see "1" in the pair (A1, B1), it means that the movement is allowed.

	A1	B1	B2	C1	C2	C3	D1	D2	D3	D4	E1	E2	E3	E4	E5	F1	F2	F3	F4	F5	F6
A1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B2	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C1	0	0	1	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
C2	0	0	1	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
C3	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
D1	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0
D2	0	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0
D3	0	0	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0
D4	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
E1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	1	0	0	0	0
E2	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	1	0	0	0
E3	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	1	0	0
E4	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	1	0
E5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
F1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
F2	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0
F3	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
F4	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0
F5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1
F6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	G1	G2	G3	G4	G5	G6	H2	H3	H4	H5	H6	I3	I4	I5	I6	J4	J5	J6	K5	K6	L6
A1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B2	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
C1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C3	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0
D1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0
E1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
E2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
E3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
E4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
E5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
F1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

	A1	B1	B2	C1	C2	C3	D1	D2	D3	D4	E1	E2	E3	E4	E5	F1	F2	F3	F4	F5	F6
G1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
G2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
G3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
G4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
G5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
G6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
H2	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
H3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
H4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
H5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
H6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
I3	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
I4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
I5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
I6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
J4	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0
J5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
J6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
K5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1
K6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

	G1	G2	G3	G4	G5	G6	H2	H3	H4	H5	H6	I3	I4	I5	I6	J4	J5	J6	K5	K6	L6
G1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
G2	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
G3	0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
G4	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
G5	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
G6	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
H2	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
H3	0	0	1	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0
H4	0	0	0	1	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0
H5	0	0	0	0	1	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0
H6	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
I3	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0
I4	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0
I5	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0
I6	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
J4	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0
J5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	1	0
J6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0
K5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1
K6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
L6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Capturing moves occur when a player "jumps" an opposing piece. This part is the main difference between Treason and the Checkers game. So, it is convenient to do a detailed explanation of this movement. Generically, the capture movement can only happen if the following conditions are satisfied:

(a) The opponent piece is in a triangle with a common vertices or common side, of the player piece.

(b) When we imagine a line between the initial position, and final position, after capture, the line of capture must have part of it inside the triangle, where the piece to capture is, and the final position must be empty. We mean with “part of a line” any part that is different of a point.

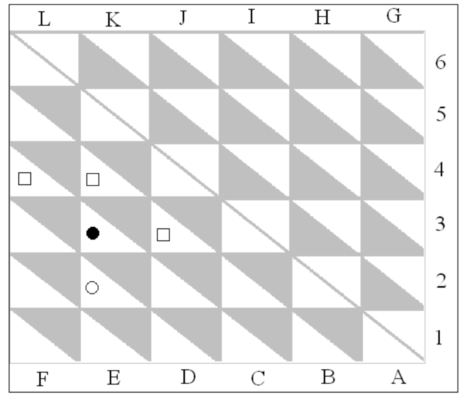


Figure 3: Capture, movement I.

Let’s start with movement I, in figure 3. White piece, in E2, can capture black piece, in E3, only if positions represented by squares in figure 3, D3, E4 and F4 are free. If so, the black piece is removed from the board and white player puts his piece in one of the three available positions.

In figure 4, we can see movement II. White piece, in F3, can capture black piece in F4, only if positions represented by squares E4 and F5 are empty. If so, the black piece is removed from the board and white player puts his piece in one of the two available positions.

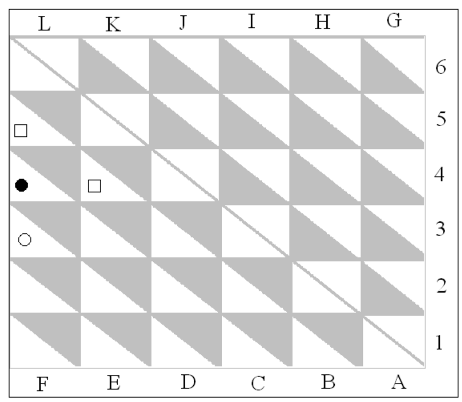


Figure 4: Capture, movement II

In figure 5, we can see movement III. White piece, in D4, can capture black piece in J4, only if positions represented by squares I4 and J5 are empty. If so, the black piece is removed from the board and white player puts his piece in one of the two available positions.

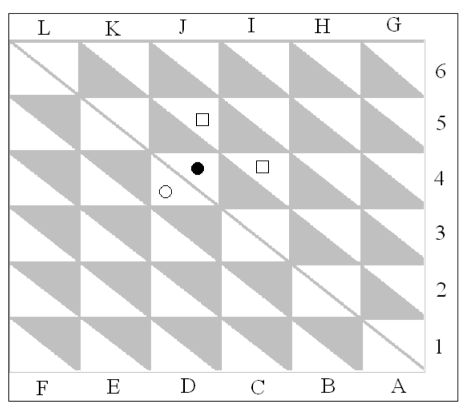


Figure 5: Capture, movement III

On a capturing move, a piece may make multiple jumps. If after a jump a player is in a position to make another jump then he may do so. This means that a player may make several jumps in succession, capturing several pieces on a single turn.

In figure 6, we can see movement IV. White piece, in D4, can capture black piece in E5, only if positions represented by squares L6, F6 and F5 are empty. If so, the black piece is removed from the board and white player puts his piece in one of the three available positions.

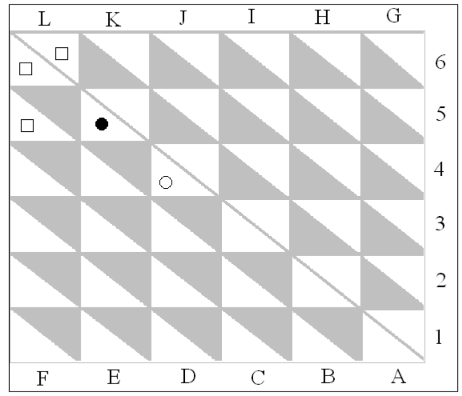


Figure 6: Capture, movement IV

In figure 7, we can see movement V. White piece, in D4, can capture black piece in K5, only if positions represented by squares L6, F6, K6 and J5 are empty. If so, the black piece is removed from the board and white player puts his piece in one of the four available positions.

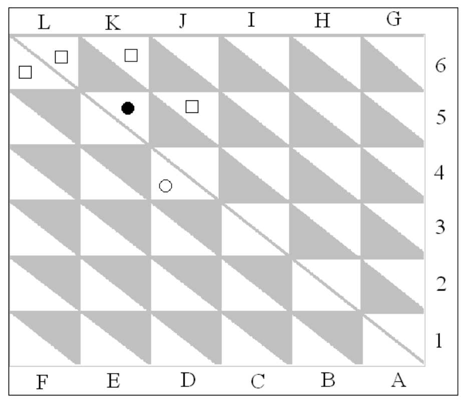


Figure 7: Capture, movement V

In figure 8, we can see movement VI. White piece, in I4, can capture black piece in J4, only if positions represented by squares K5, E5 and D4 are empty. If so, the black piece is removed from the board and white player puts his piece in one of the three available positions.

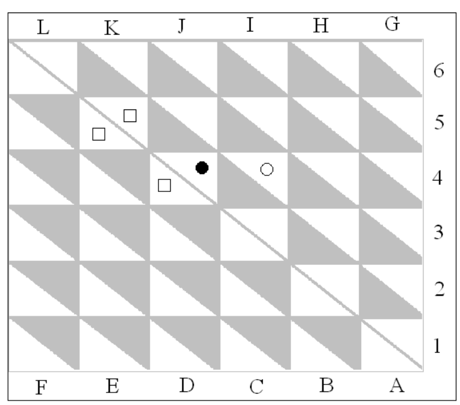


Figure 8: Capture, movement VI

When a player is in a position to make a capturing move, he must make a capturing move. This is the forced move. When he has more than one capturing move to choose from he may take whichever move suits him.

In figure 9, we can see movement VII. White piece, in I4, can capture black piece in J5, only if positions represented by squares K6, J6 and K5 are empty. If so, the black piece is removed from the board and white player puts his piece in one of the four available positions.

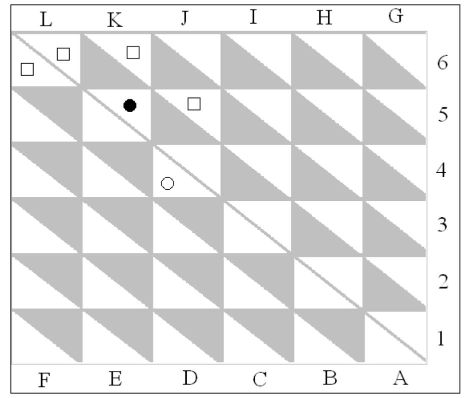


Figure 9: Capture, movement VII

There is a lot more of capture movements possible, but these examples are enough to understand the richness of strategies when we capture a piece in treason game. To finish this short explanation, we give an example, figure 10, which exemplifies that when the capture is done in different regions - lower main triangle and upper main triangle - it can be different in the final positions. In figure 10, the white piece captures the black piece and the squares represent the final positions available to the white piece.

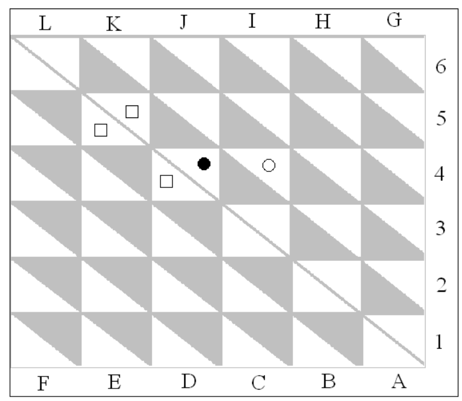


Figure 10: By-side capture

The Spy

When a piece achieves the promotion line it is transformed in a Spy. To distinguish from the other pieces we put another piece over it. The Spy must follow the capture rules allowed to a regular piece, but it can move in any direction. The Spy may now also jump in either direction or even in both directions in one turn (if he makes multiple jumps).

The Spy also has the ability to transform other spies, or regular pieces, to their side. It means that a white Spy can transform regular black or black spies in white ones. This is done using the symmetry of the board, regarding the main diagonal of the square board. That is, if the White Spy is in one position of the board, and in the symmetric position of the board, regarding the main diagonal is occupied by a black piece then it becomes white.

For example, the White Spy in position I4, can transform the black piece in D3 in a white piece, because these two positions are symmetric regarding the square main diagonal. The White Spy stays in the same position, and the black piece is replaced by a white one.
The Spy is not obliged to do this movement, and the player can use this enhance only if he wants to do so.
Any transformation counts as one move.

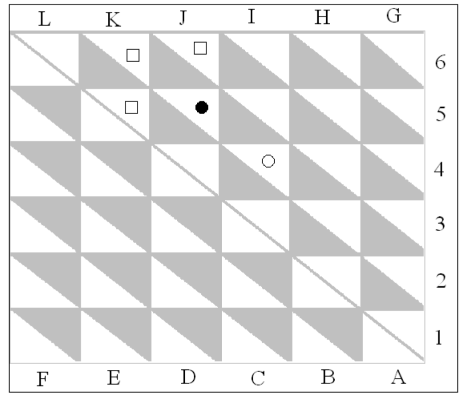


Figure 11: White Spy transforming Black

Treason Strategy

Treason is a straight-forward game in many ways. Yet, play can unfold in intricate layers. Every move opens untold possibilities and closes down untold more. Thus, it is well to keep a few strategies in mind when playing, even when it is just for fun.

First, always keep in mind the possibility of using the forced capture rule to manoeuvre your opponent into a position where he gives up two pieces for one of your own. Often a one piece advantage can make all the difference in the end game.

Second, always try to keep the lanes to the promotion row blocked to your opponent. Once either side gets a Spy, any piece in the open is highly vulnerable.

Third, move between your own pieces and your opponent in order to move adjacent to an opposing piece without loss.

Fourth, use the transformation move only if you gain an important position in the board, and the piece cannot be captured by your opponent.

Of course, these are elementary ideas to the player. To move beyond the beginner stage, this game requires a more detailed study. We hope in the future to contribute more with a deep study of the mathematics involved here, but our goal, for now, was only to present the game, and we believe this material is enough to start. Have fun with our game!

Slimetrail for the Visual Impaired

Carlota Dias
High School Matias Aires

Pedro Palhares
Institute of Education
University of Minho

Jorge Nuno Silva *
University of Lisbon

Summary

The repercussions of the visual deficits affect merely adaptive questions, known as Activities of Daily Life, as well as the capacity to access cultural information through the usual means. Therefore, a visually impaired person must optimize his/her sensory systems.

The current article begins with a presentation of ideas by several authors on the influence that visual impairment has on the development of children and the importance of game playing in stimulating that same development. Secondly, we will present some results of the research undertaken so far, concerning the adaptation of the game Slimetrail.

Visual impairment and playing games

In **Portugal** blind people are those with: complete visual absence; visual acuity inferior to 1/10, in the best eye, after conventional correction; visual acuity superior to 10°, together with a limitation in the visual field inferior or equal to 20° in each eye. As far as functional vision is concerned, every time there is a visual loss sufficiently serious, that may interfere with everyday activities, and not possible to be corrected with conventional or contact lenses, it becomes necessary to intervene with visual rehabilitation and technical auxiliaries. (Mendonça, Miguel, Neves, Micaelo & Reino, 2008)

According to Blanco & Rubio (1993) most of our judgments concerning what goes on around us, under normal circumstances, including a significant part of our knowledge of the world and ourselves, present themselves in the form of visual images. Thus, being blind keeps one from accessing a certain type of information, which makes it impossible to represent the world as being seen.

In spite of being the most important way to collect exterior information, vision on its own does not provide with total identification of the surroundings since *«(...)our interpretations of the world around us are determined by the interaction of two things: (1) the **biological structure of our brains** and (2) **experience**, which modifies that structure.»* (Smith & Kosslyn, 2009, p. 56)

The child has to understand the visual collected data so, visual perception is essential for the recognition of the surrounding environment. When the visual functions are affected by a specific

*Partially supported by Project PTDC/HCT/70823/2006

pathology that causes blindness, touch assumes a prominent role in the recognition of the world. In the case of the student with visual impairment *«touch gives information not only on the characteristics of objects, such as their shape, size, and texture, but on the functional aspects of objects, such as the possibility that they can be used as tools.(...)»* (Withagen, Vervloed, Janssen, Knoors & Verhoeven, 2010, p. 43)

According to Ochaíta (1993), difficulties of blind children on a figurative level cannot be attributed to an intellectual delay as a consequence of blindness, but instead to the problems that result from tactile perception. Those differences become null by the age of 11/12 – beginning of formal hypothetical deductive thinking.

Our perception of the world around us depends on our visual perception, the same happens in identifying and learning a mathematical game. A blind child apprehends a game mainly by touch. *«Blind people recognize objects by touch or by sound. Recognition is not dependent on a particular sensory modality. (...)»* (Smith & Kosslyn, 2009, p.70). Tactile identification is essential to learn the rules of a game to handle its pawns and identify the moves. The haptic system is the foundation of this whole process.

> *«The haptic system, unlike the other perceptual systems, includes the whole body, most of its parts, and all of its surface. The extremities are exploratory sense organs, but they are also performatory motor organs; that is to say, the equipment for feeling is anatomically the same as the equipment for doing. This combination is not found in the ocular or the auditory system. We can explore things with the eyes but not alter the environment; however we can explore and alter the environment with hands. (...)»* (Gibson, 1983a, p.99)

The child with poor vision has other difficulties when apprehending a game, and its adaptations must take dimension and colour of the material into account. *«The final area in which information must be collected is the accessibility of both materials and the presentation of information. (...)»* (Castellano, 2010a, p.17).

The variety and quantity of experiments the child was exposed to can determine the skill of understanding the environment. Some authors defended that playing games is a strategy to develop several child abilities. According to Castellano the game gets children ready for the many structured and unstructured play situations, she says *«Get your child ready for the many structured and unstructured play situations he/she will encounter in the early years.»* (Castellano, 2010b, p. 71).

Playing games can be a way of experiencing problem-solving.

> *«Play can in many ways be experienced as a problem-solving process where the child explores the environment and himself in relation to his abilities, needs, interests, and other conditions. One aspect of play that has attracted major interest in recent years is its significance in developing and strengthening the child's self-perceptions or self-concept as well as identity»* (Lillemyr, 2009a, p.6)

Design research method

The prime objective of this research is the creation of conditions that enable the participation of children and youngsters with visual handicap in the National Championship of Mathematical Games, as well as the implementation of the practice of its games in the educational system, developing gaming skill through the creation and perfection of strategies.

The method used falls in the qualitative research, mostly based on collection of data from game-playing sessions and brief interviews to the participants, and participant observation was a continuous practice. The information has been collected through direct observation in two separate stages. During the first stage a series of tests to the different game boards and respective pawns with variety in dimension, form, texture and colour were conducted. A set of rules in *braille*, enlarged, or orally explained, depending on the needs of the student, was made available. The second observation stage assumes that the student already has a good domain of the game that allows him or her to develop skills at the levels of communication and strategy creation. The investigation project is, at this time, at the end of stage one.

One aim of the study was to discover which adaptations must be made to the mathematical games (we will focus here only on the game Slimetrail) so that students with visual impairment may play it with their peers.

The sample was composed of 16 students from several schools across the country (Portugal) which were categorized in Table 1.

Table 1: Categorization of Sample

Blindness	**Poor vision**	**Male**	**Female**	**Age group**
2	10	9	3	12-15
2	2	2	2	15-18
4	12	11	5	16 students

Game Slimetrail

The game Slimetrail was invented by Bill Taylor in 1992. The original board is an hexagon and it is a highly tactical game. The National Championship version was composed by a board 7x7, 1 squared piece covered with black velcro and about 40 circular pieces in light wood.

The goal is to place the squared piece on its respective final square (a1- player one and g7- player two) or stalemate the adversary, keeping him from playing any further. It is indifferent who moves the squared piece to a goal square. For example, if player one can only move onto the goal square of player two (g7) it is still player two who wins.

The game begins with the squared piece on e5. Each player, alternatively, moves the squared piece to an adjacent empty square (moves can be vertical, horizontal or diagonal). The square where the squared piece previously was receives a circular piece. Squares which receive a circular piece cannot be occupied by the squared piece anymore.

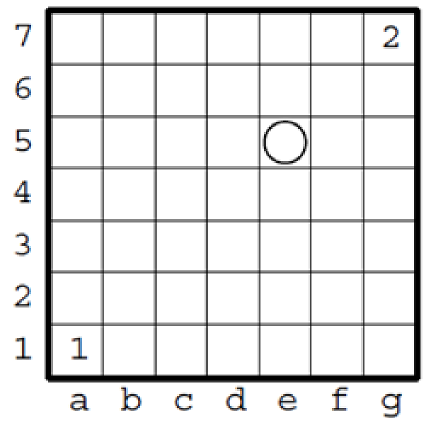


Figure 1: Slimetrail

Figure 2: Slimetrail blindness prototype

Prototype evolution for blindness

Touch is a sequential slow process which creates problems in processing/collecting data. That is why blind children have difficulties in representing the environment. So, some adaptations were made. As far as material is concerned we used a board and pieces in wood because of the material resistance. The pieces have different shapes, about 40 circular pieces and 1 squared piece.

The first prototype version was a small sized board in wood (1 squared piece and about 40 circular pieces), but there were some difficulties related to it. Without identification of the initial square, it took some time to identify the initial game position. Because of the small dimension players pushed many pieces aside. The first solution presented was a big sized board in wood, with the indication of the initial square and all the pieces were made bigger. Although most problems were solved, the students still took some time to identify the position of the squared piece. So, the squared piece was covered in velcro.

Figure 3: Slimetrail (small sized board in wood; 1 squared piece and 40 circular pieces)

Figure 4: Slimetrail (biggest sized board in wood; 1 squared piece and 40 circular pieces)

The adaptations were made without forgetting that *«Children's ability to solve problems, for example, finding out the characteristics of materials (e.g., shape, weight, etc.) will, in this approach, be regarded as a significant characteristic of children's play.»* (Lillemyr, 2009b, p.48)

Prototype evolution for low vision

Children with low vision presented different difficulties so, contrast became more important.

> *«The main visual pathways in the brain can be thought of as an intricate wiring pattern that links a hierarchy of brain areas. Starting at the bottom, the pattern of light intensity, edges and other features in the visual scene forms and image on the retina.»* (Smith & Kosslyn, 2009, p. 53)

Wood was chosen because it does not reflect light. The adaptations were similar to the prototype adaptations for blindness, except in size (smaller) because it can be entirely perceived in the visual angle, according to a student's opinion. Contrast must also be emphasized and that is why the squared piece was covered in black velcro.

> *«The faces and facets of things, the edges, corners and curvatures, are mapped into the array by virtue of a law relating the inclination of a surface to the amount of light it reflects in a given direction. But there are two other causes of structure that combine with this one - the chemical composition of substance that determines its reflectance (surface color), and the existence of cast shadows.»* (Gibson, 1983b, p.221)

The first prototype created for low vision was a board in plasticized paper, 1 red stone and about 40 black stones. In that game the stones fell out of the squares, the board reflected light (brightness) and the stones moved too easily. Thus, the board in plasticized paper was changed to a small sized board in wood. In spite of these changes, the stones still fell out of the squares. The stones had to be changed to 1 squared piece and about 40 circular pieces both in wood.

Figure 5: Slimetrail (board in plasticized paper; 1 red stone and 40 black stones)

Figure 6: Slimetrail (small sized board in wood; 1 red stone and 40 red stones)

Two more similar prototypes were made before the final version. The first one was a small sized board in wood, with 1 squared piece and 40 circular pieces, but without identification of the initial square, which made it difficult to identify the initial game position. To prevent this the initial square was covered by a different colour and a different texture. Nevertheless, students still took some time to identify the position of the squared piece in each turn. The solution was to cover it in black velcro and the final version was finished.

Figure 7: Slimetrail prototype for low vision

A bigger sized board was also tested with these students. Although some preferred the smaller one, they could all play with the biggest one.

Figure 8: Slimetrail

Conclusions

Though all students learnt the Slimetrail rules, blind students presented more difficulties than students with poor vision.

The Slimetrail final layout (for low vision and blindness) allows the students with visual impairment to compete with their peers. The average time to complete a game between two students with visual impairment is basically the same as that taken by visual students.

Future Challenges

The next challenge is to know if students with visual impairment develop winning strategies to solve playing problems.

References

Blanco, F., & Rubio, M. (1993). Perception Sin Vison In *Psicología de la Ceguera*, 3, (51-110). Madrid: Alianza Editorial.

Castellano, C. (2010a). Academics In *Getting Ready for College. Begins in Third Grade, 2*, (5-24). Charllote, North Carolina: Information Age Publishing, Inc.

Castellano, C. (2010b). Social Awareness and Social Skills In *Getting Ready for College. Begins in Third Grade, 5*, (69-86). Charllote, North Carolina: Information Age Publishing, Inc.

Gibson, J. (1983a). The Haptic System and its Components In *The Senses Considered as Perceptual System, VI*, (97-115). Westport: Greenwood Press, Publishers Group Inc.

Gibson, J. (1983b). The Visual System: Environmental Information In *The Senses Considered as Perceptual System, X*, (186-223). Westport: Greenwood Press, Publishers Group Inc.

Lillemyr, O. (2009a). What Is Play?. In O. Lillemyr. *Taking Play Seriously.1*, (3-14) Charlotte: Information Age Publishing, Inc.

Lillemyr, O. (2009b). Psychological Approaches and Play. In O. Lillemyr. *Taking Play Seriously, 5*, (47-55). Charlotte: Information Age Publishing, Inc.

Mendonça, A., Miguel, C., Neves, G., Micaelo, M., & Reino, V. (2008). Alunos cegos e com baixa visão. Orientações curriculares. Lisboa: DGIDC, DSEE, ASE.

Ochaíta, E. (1993). Ceguera y Desarrollo Psicologico In Ochaíta, E. & Rosa, A. *Psicologia de la cegueira* (111-202). Madrid: Alianza Psicología.

Smith, E., & Kosslyn, S. (2009). Perception. In E., Smith & S., Kosslyn, *Cognitive Psychology* (49-102). New Jersey: Pearson Education Inc.

Withagen, A., Velvloed, M., Janssen, N., Knoors, H., & Verhoeven, L. (2010, Janeiro). Tactile Functioning in Children Who Are Blind: a Clinical Perspective. *Journal of Visual Impairment & Blindness*, 104(1), (43 - 54).

The shape of the sound: from bird singing to western music

Carlota Simões
Center for Computational Physics
Mathematics Department and
Science Museum of the University of Coimbra

Abstract

The solid but discrete relationship between Mathematics and Music is at least as old as Pythagoras. Mathematics relates with Music at the moment of tuning an instrument, since consonant notes are produced by sounds whose frequencies have interesting mathematical relationships. The problem of tuning a musical instrument began as a problem of proportions, when the instrument was the vibrating string, but became even more interesting when the piano was created, requiring a tempered tuning discovered only after the intervention of mathematicians. However, without any knowledge of arithmetic, birds from all over the world accurately reproduce the several intervals of the diatonic scale and sing melodies in different musical scales created by man throughout the ages, from the Greek modes to the diatonic scales of our days.

Sound and length - the vibrating string and the musical intervals

The Octave

Take a loose string. The note it produces is independent of the strength used to put it in vibration mode. If we immobilize the middle point of the string, and if half of it vibrates, we will hear the octave of the first sound. If the sound of the loose string is a C, the sound of half string will be the one octave higher C. We may say that the octave is the sound representation of the fraction 1/2.

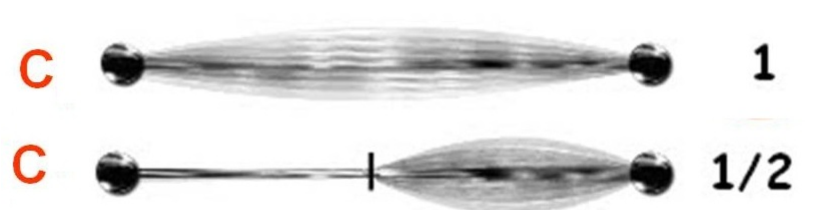


Figure 1: If the sound of the loose string is a C, the sound of half string will be the one octave higher C

Examples of the octave in music are, for instance, the first two notes of the song *Somewhere over the rainbow* (`http://www.youtube.com/watch?v=PSZxmZmBfnU&feature=related`) from the film *Wizard of Oz* (`http://www.imdb.com/title/tt0032138/`) or the animated cartoon *Dogtanian and The Three Muskehounds* (`http://www.imdb.com/title/tt0083780/`) introduction theme first intervals (`http://www.youtube.com/watch?v=VR9J2ITspyU`).

Some birds are able to sing the octave. One example is the Tecelão (*Cacicus chrysopterus*) from South America (`http://en.wikipedia.org/wiki/Golden-winged_Cacique`). The melody presented in Fig. 2 was recorded by Emerson Kaseker, at Campos do Jordão (São Paulo), and includes an octave (`http://www.wikiaves.com.br/midias.php?tm=s&t=s&s=11747`).

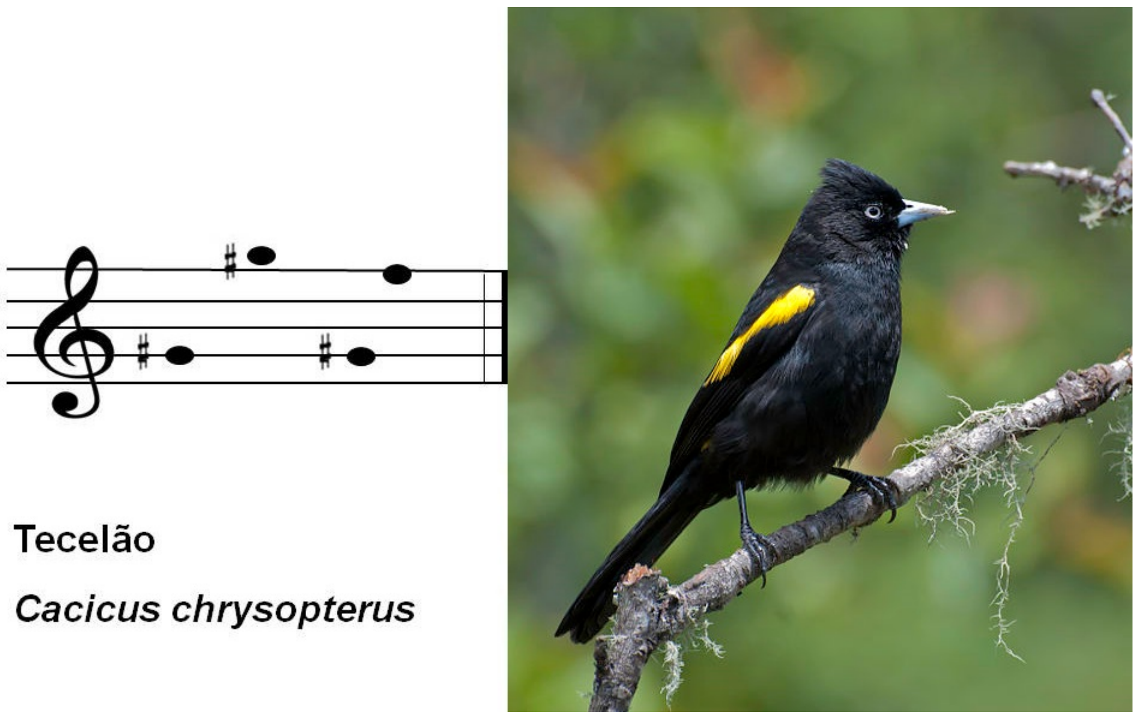


Figure 2: Tecelão (*cacicus chrysopterus*)

Credits for the picture `http://en.wikipedia.org/wiki/Golden-winged_Cacique`

The Perfect Fifth

Take the same loose string and let L be its length. To hear the perfect fifth of the first sound, we have to immobilize a point at a distance of 1/3 of L from the beginning of the string. If the sound of the loose string is a C, the sound of 2/3 of the string will be G. Since we divide the string in two parts, one being the half of the other, the sound produced at the two sides of the fixed point are related by an octave.

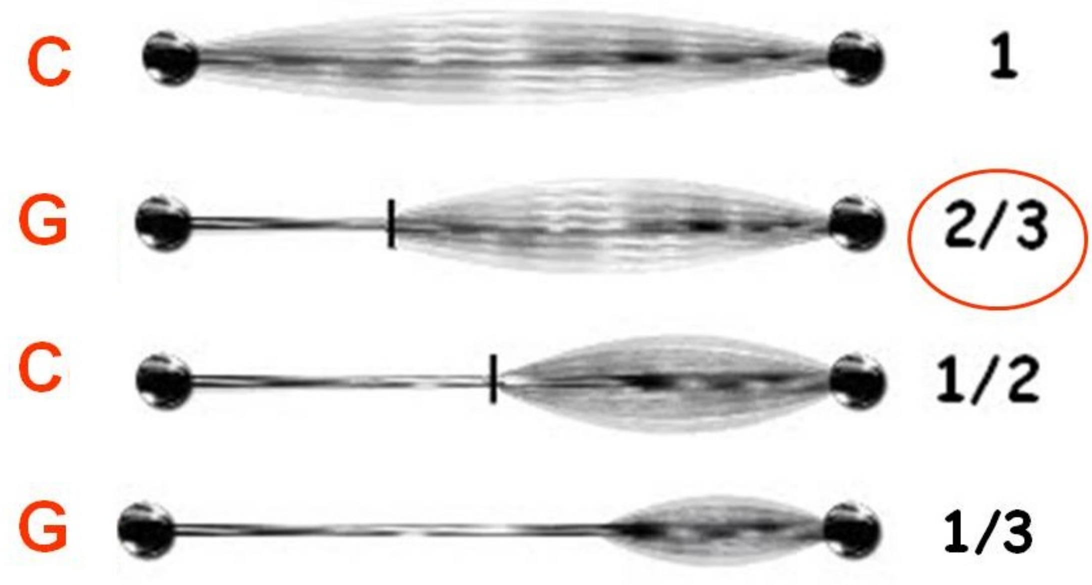


Figure 3: If the sound of the loose string is a C, the sound of 2/3 of the string will be G. The sound of 1/3 of the string will be the one octave higher G

Examples of the fifth in music are, for instance, the first two notes of the children's song *A vous dirai je maman* (known in English as *Twinkle twinkle little star*), adapted by Mozart in his KV 265 *Twelve Variations* (http://www.youtube.com/watch?v=NO-ecxHEPqI).

Many birds sing the fifth, and Cricrió (*Lipaugus vociferans*) from Amazon is one of them. The fifth is one of Cricrió's favorite intervals. Listening to this bird (http://www.wikiaves.com.br/midias.php?tm=s&t=s&s=11378) we realize how similar to its singing is Cricrió's name.

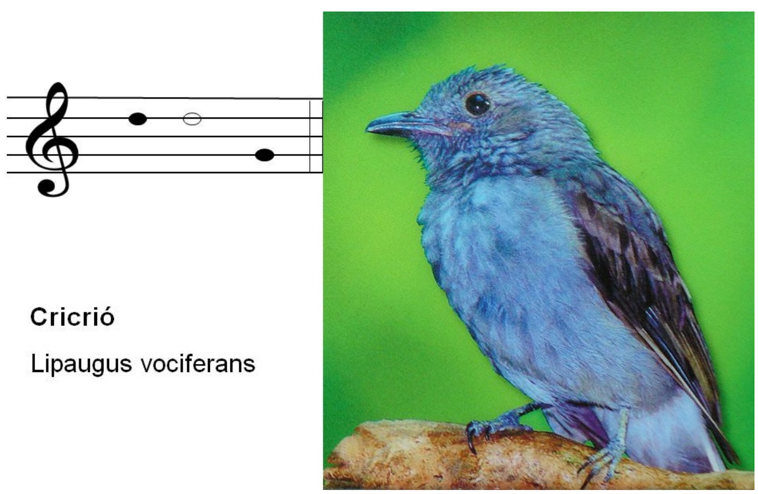


Figure 4: Cricrió (*lipaugus vociferans*)

Credits for the picture: http://en.wikipedia.org/wiki/Lipaugus_vociferans

The Perfect Forth

The perfect forth is the ascending interval from G to C. The division of the string is the same as for the fifth, but the relation between the lengths of the vibrating strings is L to 3/4 of L.

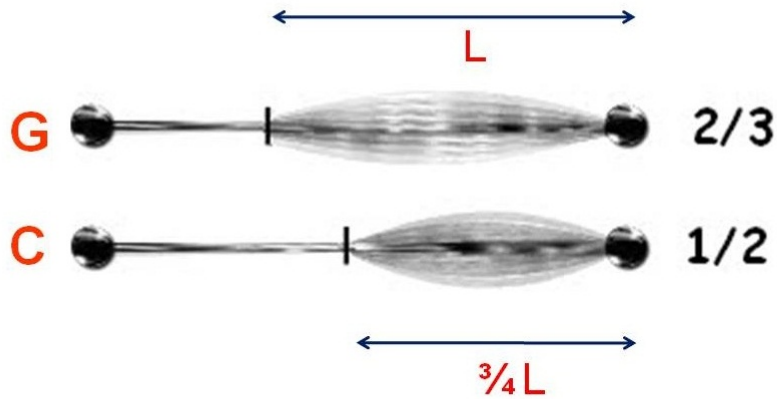


Figure 5: If the length of the longer vibrating string is L, the shorter has length 3/4 L. If the sound of the first is a G, the sound of the second will be C, and we obtain the perfect forth

A very well known example of a perfect forth in music is the first two instrumental notes of the song *Con te Partiro* of which Andrea Bocelli's interpretation (`http://www.youtube.com/watch?v=tcrfvP11Hbo`) is famous.

The Pássaro Preto (*Gnorimopsar chopi*) from South America (`http://en.wikipedia.org/wiki/Gnorimopsar_chopi`) sings a melody which includes the notes shown at fig. 6, where we can find the perfect forth (`http://www.youtube.com/watch?v=F-seW4xg8Ik`), with a rhythm similar to the aria *Habanera* of the opera *Carmen* from Bizet.

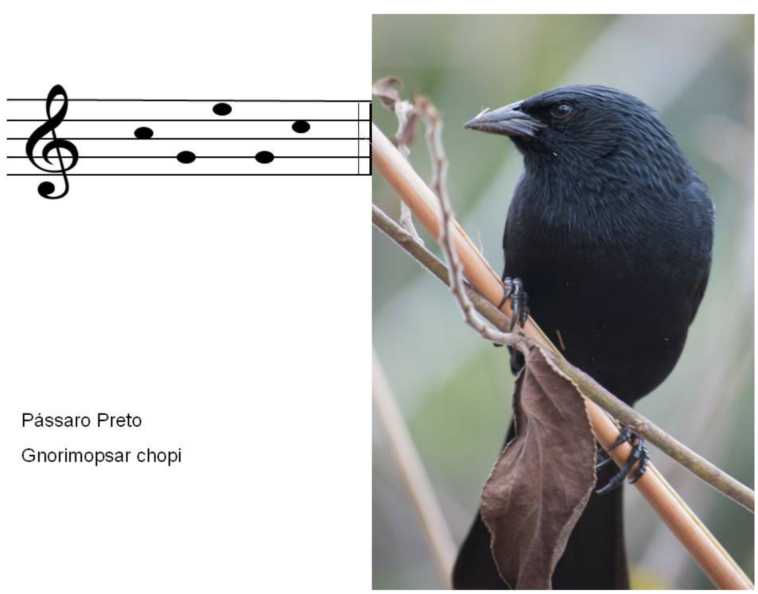


Figure 6: Pássaro Preto (*gnorimopsar chopi*)

Credits for the picture: `http://en.wikipedia.org/wiki/Gnorimopsar_chopi`

Musical scales and tuning

The Sound Wave - Frequency and Length

In physical terms, when we have a loose string and we successively immobilize one of its points, we are modifying the frequency and the period of the sound produced by the vibrating string. If a sound wave has period T, the sound wave of the octave has period T/2, the wave of the fifth has period 2/3 T and the wave of the forth has period 3/4 T. Accordingly, if a sound wave has frequency f, the sound wave of the octave has frequency 2f, the wave of the fifth has frequency 3/2 f and the wave of the forth has frequency 4/3 f.

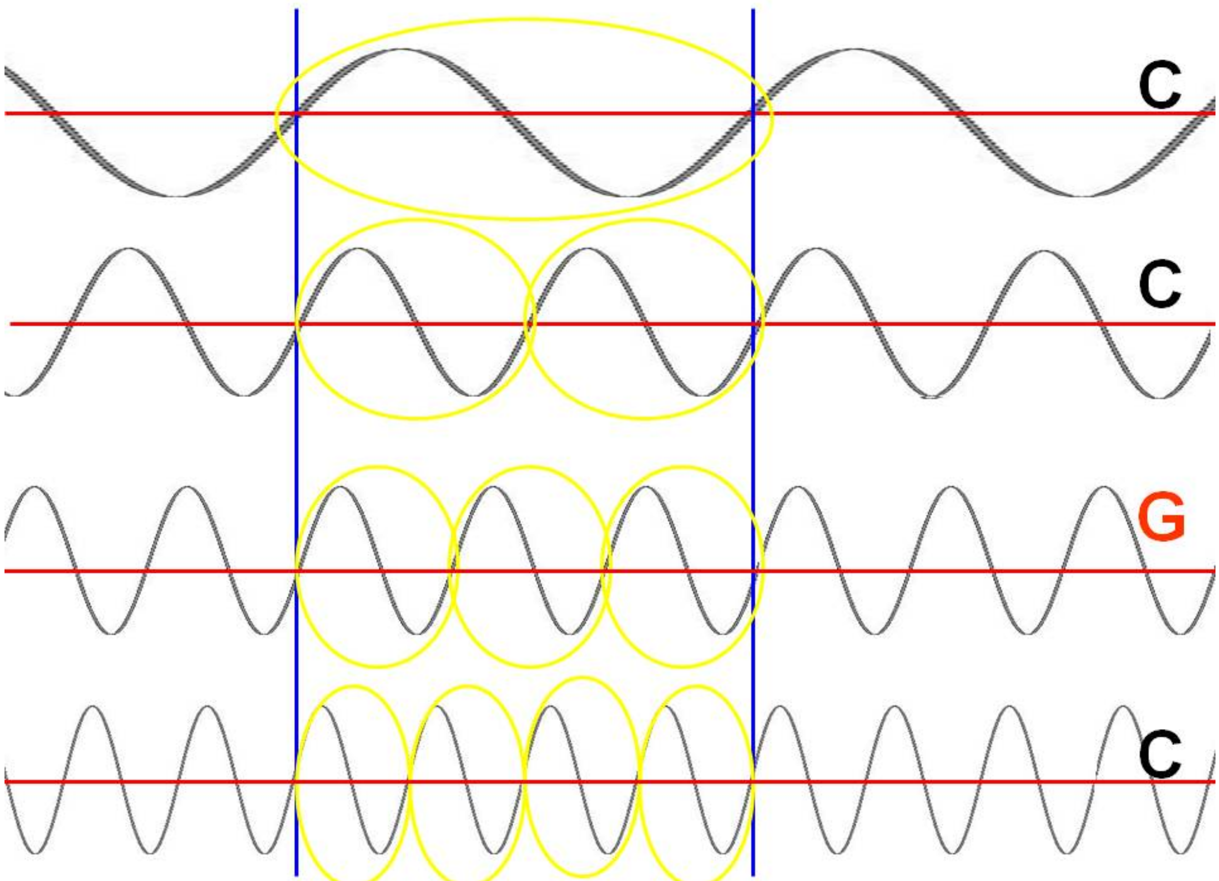


Figure 7: Relation between the sound waves of one sound, its fifth and its octaves

The sequence of harmonics

A harmonic of a sound wave with frequency f is any other wave whose frequency is a whole number multiple of f: 2f, 3f, 4f...

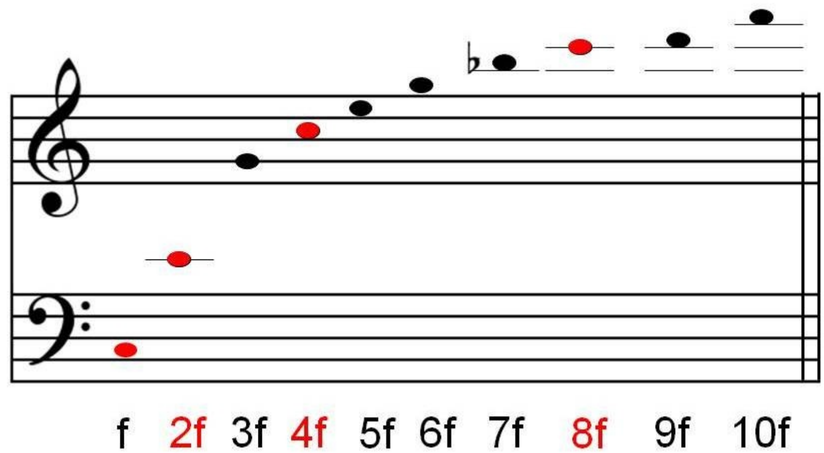


Figure 8: The sequence of the harmonics of C. (C-C) - octave; (C-G) - fifth; (G - C) - forth; (C - E) - major third; (E - G) - minor third; (G - B ♭) - minor third; (B ♭ - C) - major second, etc.

From fig. 8 we obtain one way of deducing the relation between frequencies for the several intervals. We know already that the size of an interval between two notes may be measured by the ratio of their frequencies. From fig. 8 we have the following correspondences: 2/1: octave; 3/2: perfect fifth; 4/3: perfect forth; 5/4: major third; 6/5 and 7/6: minor third, 8/7: major second, etc. With the sequence of harmonics we are able to obtain all the twelve notes of the chromatic scale (on a keyboard, ignoring octaves, the seven white plus the five black keys).

The cycle of fifths

The sequence of harmonics allows one to obtain all intervals of the scale from one single frequency f. In fact, just one single interval, the fifth, allow us to obtain the twelve notes.

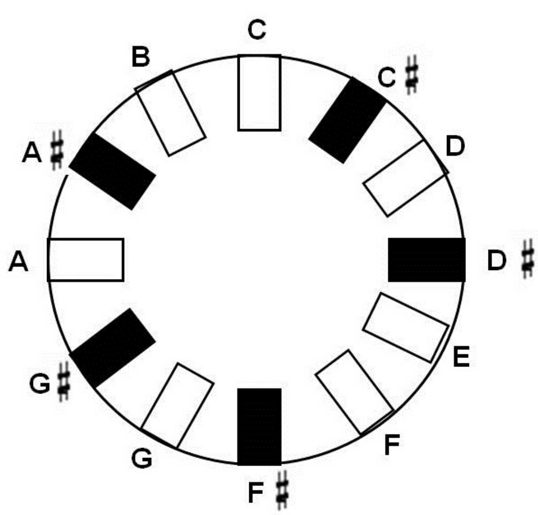


Figure 9: Imagine a round keyboard, where notes with the same name are considered equivalent, which means that the octaves are ignored

The perfect fifth interval corresponds to moving clockwise seven positions, starting from the initial key. If we start with F, we obtain the sequence F, C, G, D, A, E and B, obtaining all white keys. Proceeding with the sequence, we obtain, starting from B: F ♯ , C ♯ , G ♯ , D ♯ , A ♯ , E ♯=F.

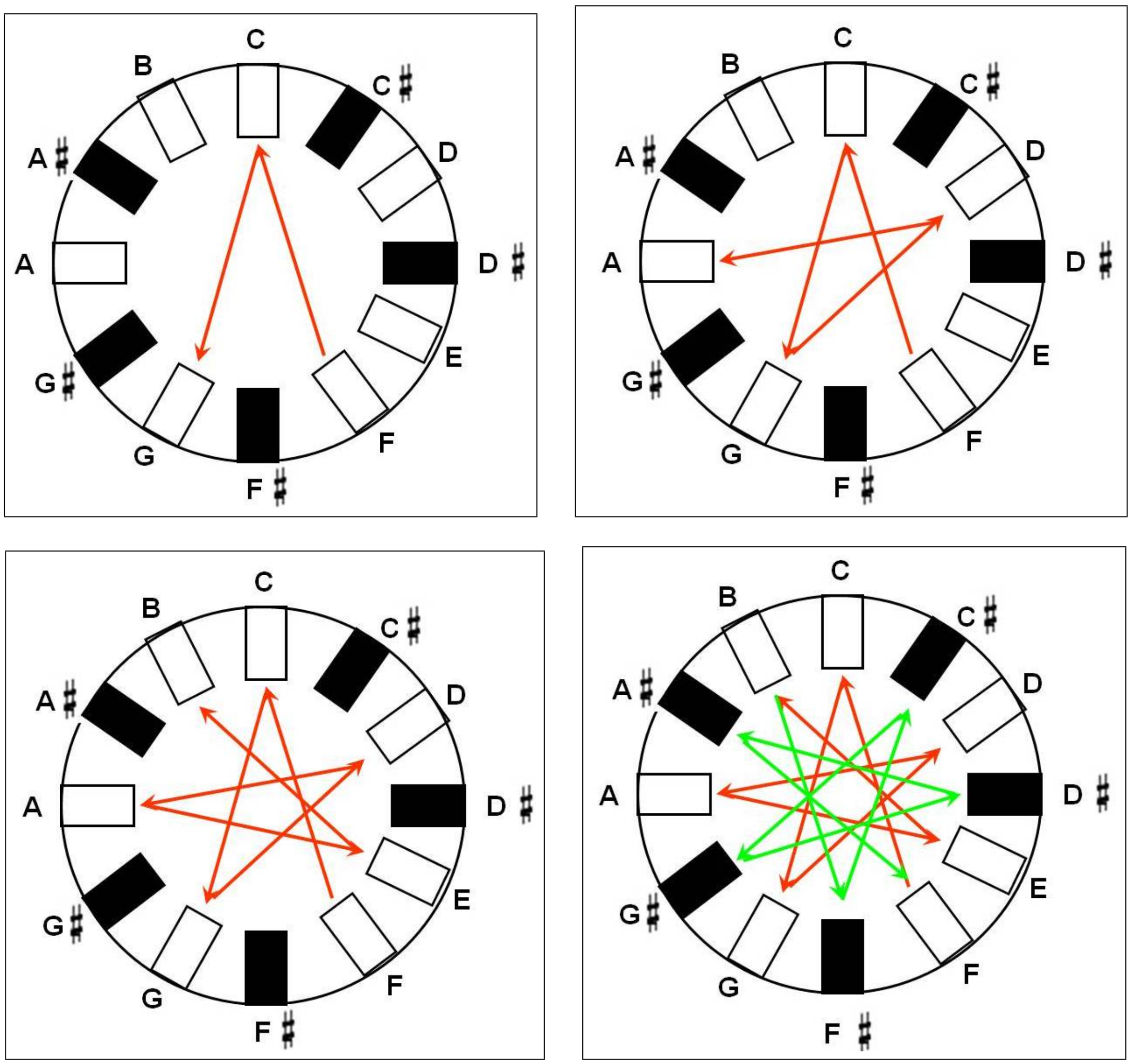


Figure 10: The fifth of F is C; the fifth of C is G. Successively we obtain D, A, E and B, all the white keys. Since the circle has 12 sides and 7 and 12 are co-prime, the sequence will reach all keys without repetitions

From fig. 10 we realize that the interval of fifth is enough to obtain all the keys of the keyboard. Since the fifth corresponds to the frequency ratio 3/2, we obtain the following correspondences to the notes of the scale.

F	f
C	3/2 f
G	$(3/2)^2$ f
D	$(3/2)^3$ f
A	$(3/2)^4$ f
E	$(3/2)^5$ f
B	$(3/2)^6$ f
F ♯	$(3/2)^7$ f
C ♯	$(3/2)^8$ f
G ♯	$(3/2)^9$ f
D ♯	$(3/2)^{10}$ f
A ♯	$(3/2)^{11}$ f
E ♯ = F	$(3/2)^{12}$ f

But here we get a contradiction. Supposedly, while closing the circle, we should return to an F, several octaves higher than the initial one, which means that the last frequency should be a power of 2. But there is no n such that $(3/2)^{12} = 2^n$.

It's very easy to find other contradictions. Let's use the proportions we found in the sequence of harmonics of fig. 8. The frequency of the central A is 440 Hz, so the next A has frequency 880Hz. On the other hand, since A - C is a minor third, the frequency of C is 6/5 of 440Hz; since C - G is a perfect fifth, the frequency of G is 9/5 of 440 Hz; since G - A is a major second, the frequency of A should be 72/35 of 440 Hz, equal to 905 Hz, quite different from 880 Hz.

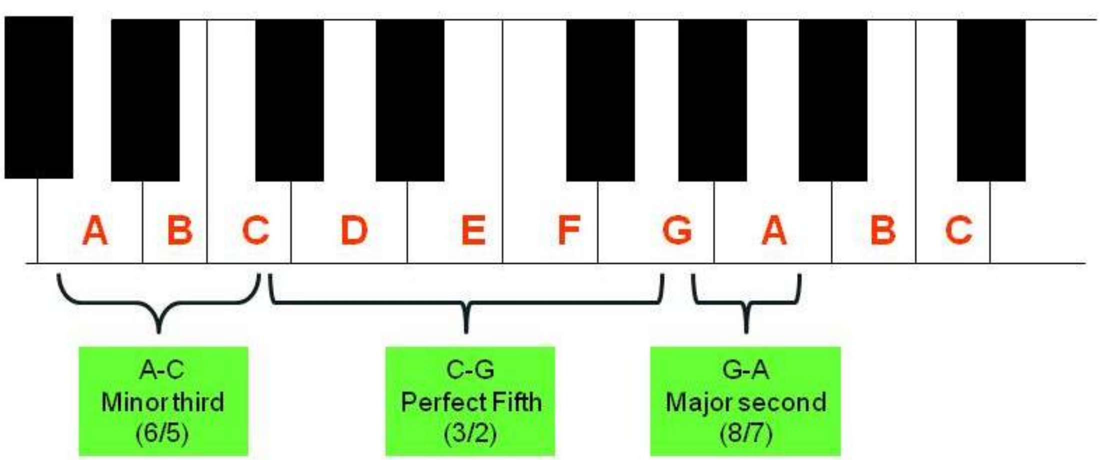


Figure 11: Partition of the octave A - A in three different intervals

Equal Temperament

In reality, the 12 different notes of the keyboard we know today are not tuned in any natural way. There were many proposals during the XVI and XVII Centuries, both from musicians and mathematicians, for a compromising *temperament*. A temperament is a method for choosing the frequencies allocated to each of the twelve notes of the scale on a keyboard. Early theorists of temperament for the 12 notes wanted to save as much as possible the natural scale intervals. Zarlino's method for tuning a keyboard (1558) maintains the natural white keys and the black keys are tuned afterwards. This is a good solution for tones like C major, G major, or A minor, which uses few black keys, but it creates problem for scales like G ♯ Major which uses all the black keys of the keyboard. To avoid this inconvenience, and thanks to the persistence of Bach and Rameau, the equal temperament was adopted. Equal temperament's theory is due to the German Werckmeister (1691). This method, entirely artificial, divides the octave into twelve half-tones in such way that, if the first note has frequency f, the second has frequency $\sqrt[12]{2^1}$ f, the third has frequency $\sqrt[12]{2^2}$ f, and the n-th note has frequency $\sqrt[12]{2^{n-1}}$ f, for n = 1, ..., 12. The only tuned notes are the first note and its octave. Curiously enough, in the 1570's, one century before Werckmeister, Vincenzo Galilei, father of Galileo Galilei, proposed a solution with equal half tones. The factor proposed by Galilei was $18/17 \approx 1,058823$, quite close to $\sqrt[12]{2} \approx 1.05946$. The equal temperament has the disadvantage of not giving any natural interval beyond the octave, but only this system allows the variety of modulations that characterize all the music after Bach. In the eighteenth century, Bach proved that a tempered harpsichord could play in all major and minor tones.

Do birds know the musical scales?

An interesting exercise for bird lovers with surprising results is to look for similarities between bird singing and human music. Maybe a bird song is a familiar melody; maybe a certain bird seems to use only the notes of a certain musical scale; maybe other birds sing musical intervals uncommon for us, humans. We present a few examples showing how inspiring can be this activity.

Bird singing and diatonic scales

Familiar melodies

The Undulated Tinamou (*Crypturellus undulatus*) (`http://en.wikipedia.org/wiki/Undulated_Tinamou`), known in Brazil as Jaó, repeatedly sings the melody of fig. 12 (`http://www.youtube.com/watch?v=O2JSn6xG-qg`) which look pretty much like the song *The Man I Love* (Gherswin), especially in the interpretation of the Brazilian singer Caetano Veloso (`http://www.youtube.com/watch?v=0FD0li0F2sg&feature=related`).

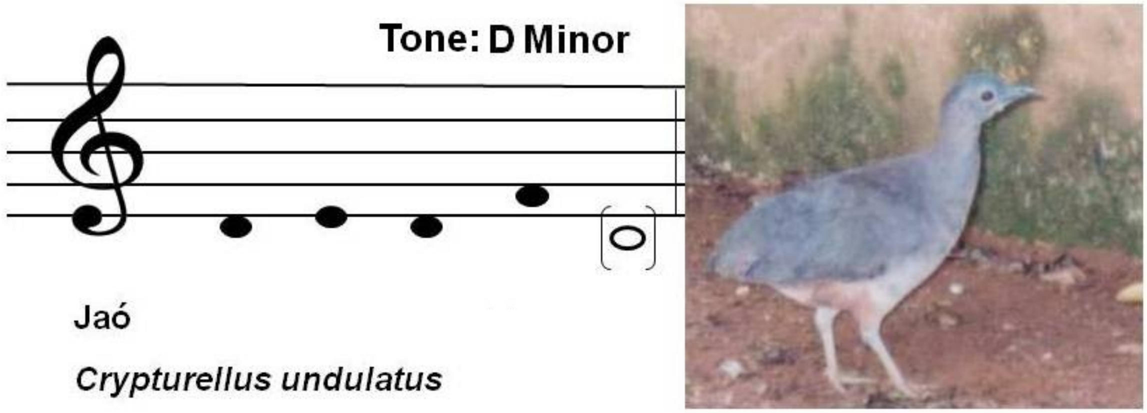


Figure 12: Jaó (*criturellus undulatus*)

Credits for the picture: `http://en.wikipedia.org/wiki/Undulated_Tinamou`

The Tropical Mockingbird (*mimus gilvus*), (`http://en.wikipedia.org/wiki/Tropical_Mockingbird`), known in Brazil as Sabiá da Praia, sings many different songs, being an imitator of other birds as well. The melody of fig. 13 (which is the beginning of the singing recorded here: `http://www.youtube.com/watch?v=c56PXOTQjwM`), seems to be in C Major and recalls the famous slogan of the French Revolution, *Ah! Ça Ira* (in English "Oh, it'll be fine"). We can hear Edith Piaf's version of this song (1953) following the link `http://www.youtube.com/watch?v=rauZMrXqRu0`.

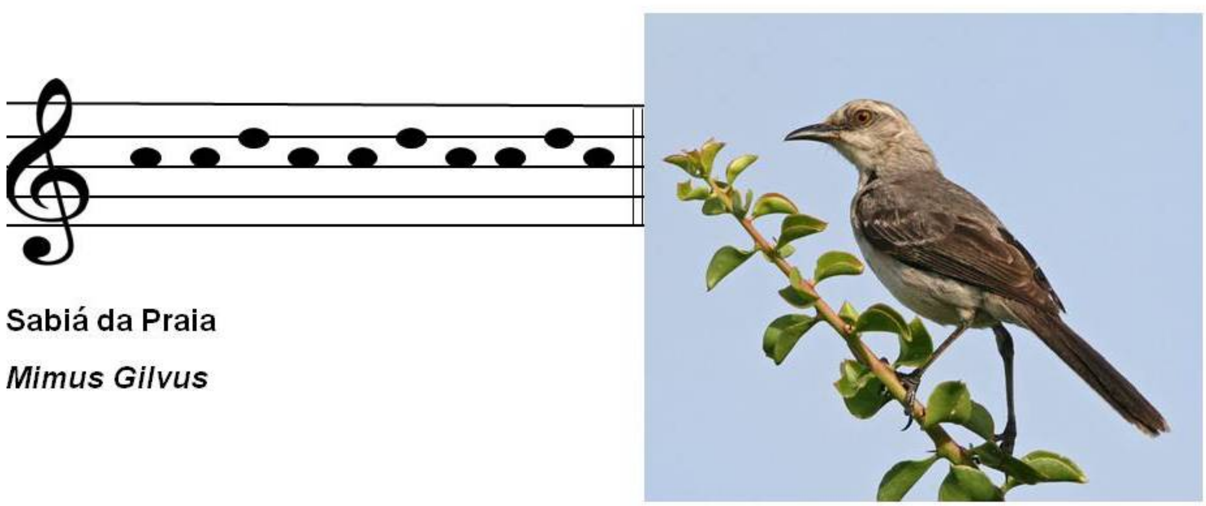


Figure 13: Tropical Mockingbird (*mimus gilvus*)

Credits for the picture: `http://en.wikipedia.org/wiki/Tropical_Mockingbird`

Bird songs and scales

The Sabiá da mata (*turdus fumigatus*) (http://en.wikipedia.org/wiki/Cocoa_Thrush) sings the melody of Fig. 14 (http://www.youtube.com/watch?v=2ILn3OhDXQU) which seems to be in the tone of C Major.

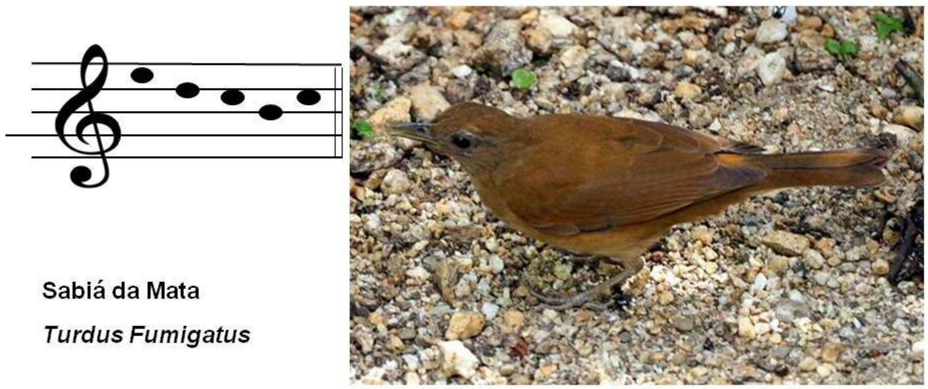


Figure 14: Sabiá da mata (*turdus fumigatus*)

Credit for picture: http://en.wikipedia.org/wiki/Cocoa_Thrush

The Tico-tico (*Zonotrichia capensis*) can be found in a large part of America, from south Mexico to Terra del Fuego. We can hear the melody of fig. 15 both by a bird in captivity (http://www.youtube.com/watch?v=5kpnabDypzQ&feature=related) and by a free bird (http://www.youtube.com/watch?v=k0aoKqiphOU&feature=related). We may abusively say it is a melody in G minor natural, but it could also be a melody in the Aeolian Gregorian mode, since this is the natural minor scale (http://en.wikipedia.org/wiki/Musical_mode#Aeolian_.28VI.29).

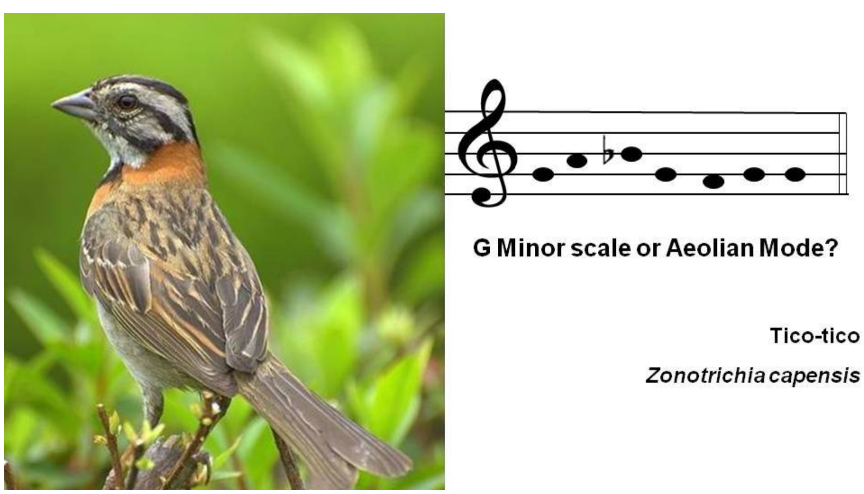


Figure 15: Tico-tico (*Zonotrichia capensis*)

Credit for picture: http://pt.wikipedia.org/wiki/Tico-tico

Unusual intervals

The tritone interval is composed by three whole tones, or six half tones. The triton is also the perfect forth plus one half tone, so we may call it augmented forth. In fig. 16 we can see how the tritone divides the diatonic scale in two equal intervals. Examples of tritone in music are quite rare, since for many centuries this was a forbidden interval, even called *diabolus in musica* (devil in music).

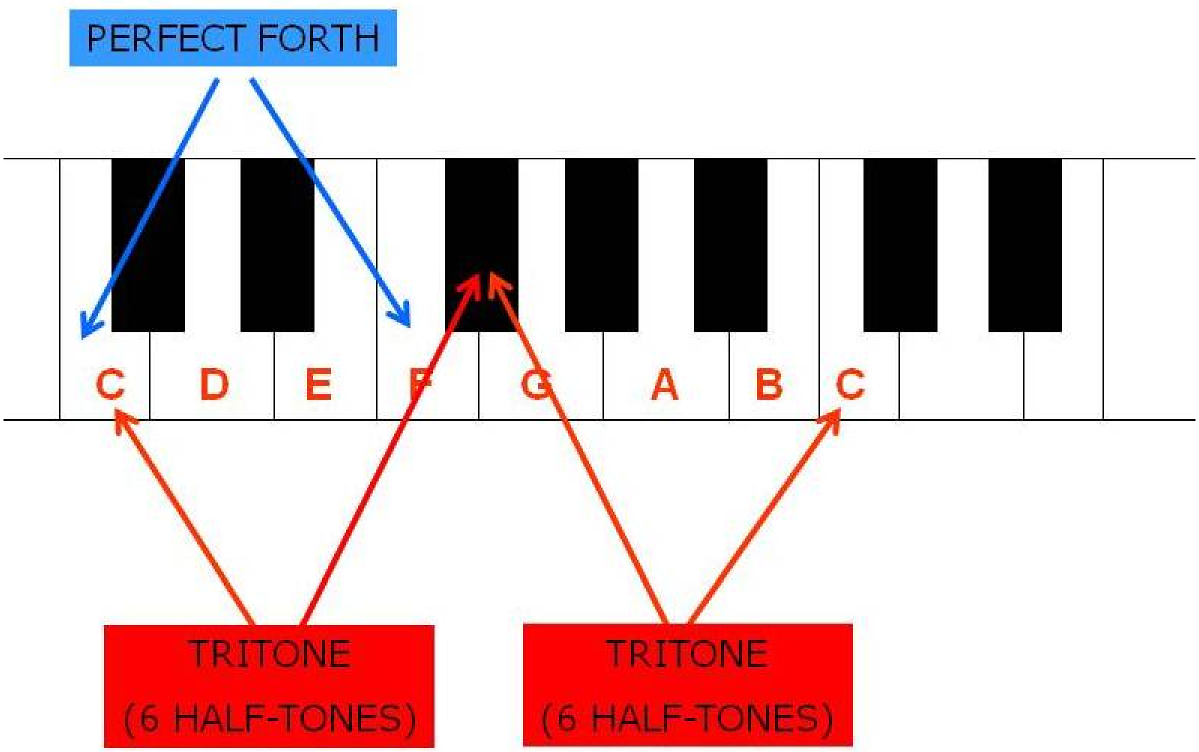


Figure 16: The diatonic scale split in two tritones

We can find a tritone in the song *Maria* (Leonard Bernstein) from the musical *West Side Story*, the first two notes of the sentence *"Maria, I just met a girl named ..."*, by Tony (`http://www.youtube.com/watch?v=VpdB6CN7jww`). Another popular example of the tritone is the first two notes of *The Simpsons* theme song (`http://www.youtube.com/watch?v=Xqog63KOANc`).

The Musician Wren (*Cyphorhinus aradus*), called in Brazil uirapuru verdadeiro (`http://en.wikipedia.org/wiki/Musician_Wren`), lives in the Amazon Rainforest. It sings wonderful and varied songs, and never repeats itself. One example of its chant is here `http://www.youtube.com/watch?v=1ptgWSpK_RU` and there are plenty to find in the internet. The melody presented in Fig. 17 can be found here (`http://www.wikiaves.com.br/uirapuru-verdadeiro`). It is astonishing the amount of different forth's it presents. We hear perfect forth's, augmented forth's (or tritones) and even diminished forth's (a diminished forth is a perfect forth with one half tone less).

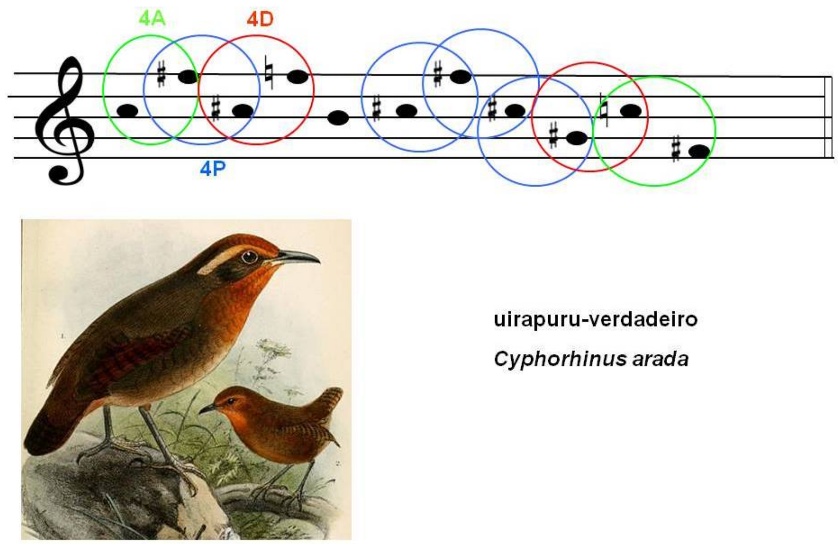


Figure 17: Uirapuru verdadeiro (*Cyphorhinus aradus*). The perfect forth's are in blue, augmented forth's in green and diminished forth's in red

Credits for the picture: `http://en.wikipedia.org/wiki/Musician_Wren`

The half tone is the smallest musical interval usually used in western music, despite de fact that intervals smaller than the half tone are common in Indian or Arabic music.

The uirapuru-veado (*Microcerculus marginatus*) (`http://pt.wikipedia.org/wiki/Uirapuru-veado`) is able to sing intervals smaller than the half tone are. We can listen to this bird singing a melodic line and going higher in so small intervals that it is difficult for human ears to identify its progression (`http://www.youtube.com/watch?v=z7M4oMpaLt8`).

Figure 18: Uirapuru veado (*Microcerculus marginatus*)

Credits for the picture: `http://pt.wikipedia.org/wiki/Uirapuru-veado`

Conclusion

That Mathematics and Music are related is not a novelty, but we tend to forget the link between Mathematics and Nature. Not so organized or predictable, Nature seems further away from exact sciences. But when we are dealing with recreational mathematics, it's good to be open-minded, and look for any source of inspiration in search of mathematical relations, patterns and analogies. Listening carefully to the birds may help us to be aware of details, with the advantage of actually enjoying their sometimes wonderful singing.

References

Benade, Arthur (1976). Fundamentals of Musical Acoustics, Dover Publications, New York.

Hartshorne, Charles (1973). *Born to Sing - An Interpretation and World Survey of Bird Song*, Indiana University Press, Bloomington.

Kroodsma, Donald E. & Miller, Edward H., ed. (1982). *Acoustic Communication in Birds*, Academic Press, New York.

Levitin, Daniel J. (2006). *This is Your Brain in Music*, Dutton/Penguin, New York.

Simões, Carlota (1999). "A ordem dos números na música do Séc. XX", *Revista Colóquio Ciências*, n. 24, pp. 48-59, Fundação Calouste Gulbenkian.

Simões, Carlota (2002). "Mathematical aspects in the Second Viennese School of Music", *Mathematics and Arts: Mathematical Visualization in Art and Education*, pp. 105-117, Editor. Claude P. Bruter, Springer-Verlag, Berlin.

Wikiaves. `http://www.wikiaves.com.br/`

Wikipedia. `http://en.wikipedia.org/`

Juggling: Theory and Practice

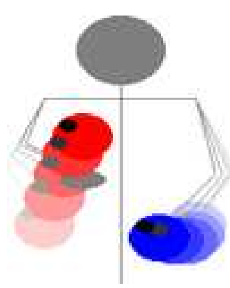
Colin Wright *

Juggling has fascinated people for centuries. Seemingly oblivious to gravity, the skilled practitioner will keep several objects in the air at one time, and weave complex patterns that seem to defy analysis. As the first known depiction of jugglers dates back nearly 4000 years, it's hard to imagine there's anything new to learn.

But a lesson we've learned from Martin Gardner is that there's always something new to learn, always something new to discover. So let's start with a quick review of classical juggling, and then see what new things we found, partly by accident, partly by hard work, and mostly because with mathematics we can see things that are otherwise hidden. We start by describing briefly the classic juggling patterns.

The Standard Pattern

The most common misconception is that when we juggle, the balls go round in a (highly elongated) circle. Juggling the balls in a cycle like this requires that every time a ball is thrown it must be handled by each hand (and therefore at most twice for all but the exceptionally gifted). In particular, the hands do different things. One hand catches the ball and shunts it over, the other hand receives the ball and then launches it into the air.



Exercise: ignoring air resistance, what are the paths of the balls? Warning: It's not a parabola.

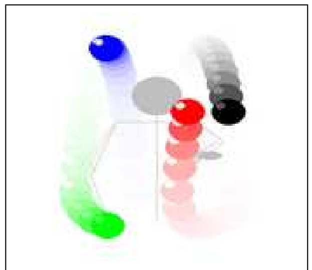
Anyone who tries this with two balls will launch with their dominant hand, showing clearly that when juggling the **throw** is critically important, not the catch. If the throw is perfect, the catch will take care of itself.

So what happens if we ask that each hand does the same thing, and each ball does the same thing (as each other ball, not the same as the hands. That would be silly).

Firstly, each ball must be thrown in turn. If not, then one ball must have overtaken another, so their throws aren't the same. If we're juggling n balls, and each ball is thrown in turn, it then becomes clear that there are two distinct cases: either the number of balls is divisible by the number of hands, or it isn't.

*Colin Wright took his B.Sc. at Monash University, Australia, and his Ph.D. at Cambridge University, UK, both in Pure Mathematics. These days he is Director of Research at a company which makes maritime surveillance equipment, still finding time to give presentations all over the world on "Juggling - Theory and Practice."

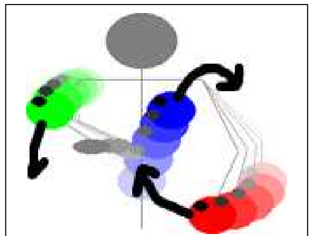
Concentrating on the case of the juggler with exactly two hands, that give us two distinct cases: an even number of balls, and an odd number of balls.



Four ball fountain

If we're juggling an even number of balls, and the balls are all thrown in turn, each ball will have to stay in the same hand. The pattern then becomes half the balls in each hand and is referred to as the "Fountain" pattern. (Pictured here at right.) Although often regarded by non-jugglers as somehow being akin to cheating, the analysis above shows that the non-changing of hands is required by the condition that all throws be the same.

Equally, if we're juggling an odd number of balls, each ball must be thrown by alternate hands. This leads to a "Lazy-Eight" pattern known (in English) as the "Cascade". Again, with an odd number of balls, the condition that every throw be the same requires us to have each ball changing hands.



Three ball cascade

These we call the "Standard Pattern" for a given number of balls. Other conditions are often imposed for basic patterns, most commonly that throws (and hence catches) occur in a metronomic rhythm, and throws are made inside shoulder width, and catches are made outside shoulder width.

Juggling in theory

So some simple analysis can tell us things about juggling patterns that we might otherwise not realise. One of the earliest published examples of this came from Claude Shannon, the father of Information Theory. The "Shannon Juggling Theorem" says that when juggling the standard pattern we have $b(d+e) = h(d+f)$ where b is the number of balls, d is the dwell time (the time a ball spends in the hand), e is empty time of a hand, h is the number of hands, and f is the flight time of a throw. (Sloane & Wyner, 1993)

But what about non-standard patterns? There are literally infinitely many possible variations on a theme. The throws and catches can be made in many different places, the timing can be varied, balls can be carried through and around other balls, multiple balls can be thrown and caught together, and so on.

The possible variations are so great both in style and detail that it is unsurprising that, despite thousands of years of history, until the mid-1980s there was no notation for juggling patterns. Even then the vast array of possibilities seemed to make the task of devising a notation impossible.

Most people who start juggling want to learn four, then five, and so on. Here is the secret to learning to juggle five - don't practice five! Instead, practice each required skill separately. For each skill - hand speed, throw angle accuracy, throw height accuracy, rate of throw, etc. - find a simpler trick that requires that skill and practise that trick. After finding and mastering a trick for each separate skill, putting them together becomes achievable in a much shorter time than trying to acquire all the skills ~~symultaneously~~ ~~simultaniously~~ ~~similtayneusly~~ at the same time. And it's more fun.

Simplifying juggling

To make progress we simplify the domain of discourse. Specifically, we assume from here on that throws happen to a metronomic beat, and that the throws and catches happen just as for the standard pattern. Further, we assume that we only throw one ball at a time, and we only catch one ball at a time.

We're now left with very few options for finding variations in our juggling. Specifically, when we throw a ball, the time it spends in the air is quantised, because it has to come down at one of the prescribed times for catches. Further, once we know *when* it comes down, that controls how *high* it goes, and *where* it comes down. So we describe each throw by a single number - the time it spends in the air. However, since we don't know what proportion of time the hands spend full (or empty), it makes our task easier to think not of the catch, *but of the next throw*. Now we can see that because throws are separated by a whole number of beats, each ball spends a whole number of beats in its journey from one throw to the next. Each throw can be described entirely by this single number.

The magical thing about this number is that when we're juggling three balls in the standard pattern, each ball is thrown every third time, so the number to describe the throw is a 3. And there's nothing special about 3. Whenever we juggle the standard pattern for n balls, each ball is thrown every n^{th} throw.

Back in the mid 1980s it was realised (Solipsys1) that some of the well-known juggling tricks could be described completely just by the appropriate string of numbers to describe the throws. Obviously when juggling the standard three ball pattern we can write ...3333... and for 8 balls we can write ...8888... and so on, but there is a well-known trick with four clubs. Normally juggled with double spins, throw one club high with a triple spin, and the next club low with a single, each club changing hands. Each club drops into the slot vacated by the other, and the pattern then continues as if nothing happened. Much less impressive when done with balls, it is a useful exercise to practise the exact height required for five ball juggling. The high throw will next be thrown five beats later, so is described as a five. The low throw is a three, so we can describe a single instance of this trick as ...444_53_444...

Another well-known four ball trick is make two consecutive throws as if juggling five, pause, and then restart. This can be described as ...444_552_444... (Exercise: why is a momentary hold described as a 2?)

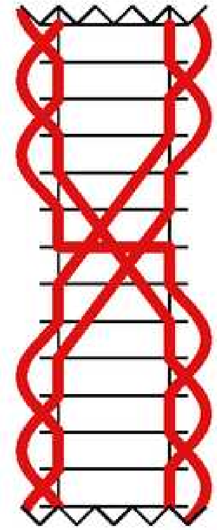
Another variation is to throw all four balls as if juggling five. Of course, after the first four throws we've run out of balls, but if we wait for a beat all the balls come down in order and we can restart our four ball pattern. We write this as ...444_55550_444... It's no surprise that for that moment when we don't have a ball we describe it as a 0, although we shall shortly see that this raises some interesting questions.

Putting it all together ...

Collecting these different tricks and writing them one above the other, putting at the top the uninterrupted four ball fountain, we end up with this:

... 4 4 4 **4** 4 4 4 ...
... 4 4 4 **5 3** 4 4 4 ...
... 4 4 4 **5 5 2** 4 4 4 ...

... 4 4 4 **5 5 5 5 0** 4 4 4 ...



Space-time diagram for 5551

The pattern was almost impossible to see when we first wrote these down, but leaving the gap makes it unmistakable. The pattern ... 444 **5551** 444 ... is clearly missing, and based on the sequence, clearly should be a juggling trick.

But it was a trick we didn't know.

From a four ball fountain throw three balls as if juggling a five ball cascade. Now you have one ball left - DON'T THROW IT! Zip that ball across into the otherwise empty hand. Now instead of waiting for a beat, you can carry on immediately.

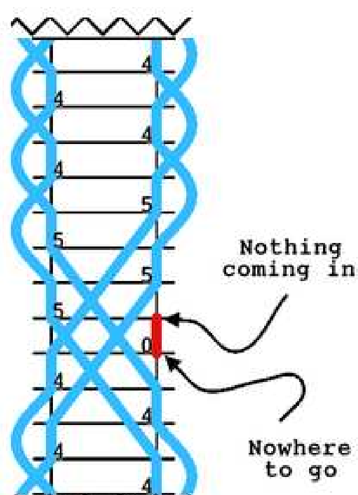
Do this constantly, and suddenly it feels a lot like five balls. Three out of every four throws is a 5-ball throw, and the pattern is there, in the air, with just a flicker for a missing ball every fourth beat. Superb practice for 5, and enormously easier as it's only four.

An entirely new juggling trick, discovered through mathematics.

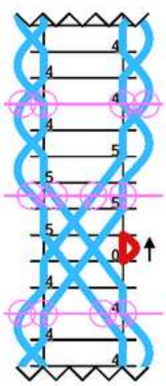
A mystery emerges ...

We've shown that some juggling tricks can be described by sequences of numbers, and that by following patterns we can find previously unknown tricks. Not all sequences make valid juggling tricks, but space does not permit investigation of that particular aspect.

There is another question that emerges, however, when we look at the physical reality. We do, after all, have to hold the ball between catching and throwing. In our previous Space-Time diagram we've made the simple assumption that the hands are full for exactly half the time, and we can see that the throws that come back to the same hand are four beats from throw to throw, three beats in the air, giving a hold time of one. The high throws that change hands are fives, and they spend four beats in the air. The zip across is no time in the air, one beat in the hand, and therefore its "Cycle Time" - the time to the next throw - is 1. All this is just as we might expect.



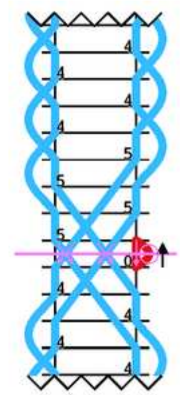
But what about the 0 in 55550? Every other number is the time from throw to throw, and the time in the air is one less. If we follow that pattern, the 0 should give an air-time of -1. We have predicted the time-travel of a juggling ball. How can that possibly work??



If the hands are full for half the time we end up with a ball in the right hand that has come from nowhere, and has nowhere to go. Clearly we should have the ball go back in time to become itself, just as required.

If we draw a horizontal line on our diagram it's a single instant of time, dividing past above from future below. In a sense it's a photograph, freezing the action and seeing where things are. The diagram here at left has several photographs, each showing where all four balls are.

In each case there's a ball in the hand and balls in the air, always exactly four of them. Which is right and reasonable, as we are juggling four balls. By the conservation law of juggling equipment we should always have four balls.



But look at the photograph in the diagram on our right. Here we have four balls in the air between the hands, and another ball in the right hand. Clearly there's something strange happening. But wait! There's more! There's also a ball going backwards in time. That must count as a negative ball, to bring our count back to the required four.

It's an anti-ball!

We can think of the "catch" (where the ball comes from the future) as the mutual creation of a ball/anti-ball pair, and the throw back into the past as the mutual annihilation. Thus we have confirmed the view in modern physics that an anti-particle can be thought of as a particle going backwards in time: a positron is an electron going backwards in time, an anti-proton is a proton going backwards in time, etc. More, since a photon is its own anti-particle it doesn't know whether it's coming or going, but since it travels at the speed of light, Einstein tells us time is stopped.

But $E = mc^2$, so where does the energy come from to create a ball/anti-ball pair? Just as there's a quantum uncertainty principle between position and momentum, there's also a quantum uncertainty principle between energy and time. We know exactly when the throws and catches are happening, so we have a very small uncertainty in time and we can borrow from the quantum uncertainty in energy to create a virtual ball/anti-ball pair.

In truth, the anti-ball can be thought of as subtracting a ball from where we expect one, leaving us with an empty hand when our assumptions would normally require a ball.

And in conclusion ...

It doesn't end there. Now there are notations for hand movements, timing variations, patterns involving more than one juggler. We have arithmetic methods for determining whether a given sequence can be juggled, and algorithms for producing all possible juggling sequences with any number of balls. Work continues to make these newer notations simpler, cleaner, and more useful. But the real bonus is that this material is being used as a vehicle to bring the excitement and enthusiasm of recreational mathematics to thousands each year, year on year.

The juggling is fun, but the maths, as one student said to me, is "funner".

Footnotes / References

Sloane, N. J. A., & Wyner, A. D., (Eds.). 1993. A draft paper for Scientific American is included in *Claude Elwood Shannon Collected Papers*,. New York, IEEE Press, pages 850-864.

Solipsys1. `http://www.solipsys.co.uk/new/Juggling.html`

Solipsys2. `http://www.solipsys.co.uk/new/ColinWright.html`

Wikipedia. `http://en.wikipedia.org/wiki/Siteswap`

CECM. `http://www.cecm.sfu.ca/organics/papers/buhler/paper/html/paper.html`

How High the Moon

Colin Wright

Some time ago while drifting off to sleep it seemed like a bunch of stuff I knew all tied up together into a neat bundle.

Then I woke up.

Surprisingly, it all still worked! Here are the results.

We're going to compute the distance to the Moon using a few well-known facts, a few simple observations, a pendulum, and a stopwatch. Pretty much everything here was known to Isaac Newton in the late 1600's, and it's even been suggested that he performed pretty much exactly these calculations.

Maybe, maybe not. Let's just see what we can do with some really elementary reasoning.

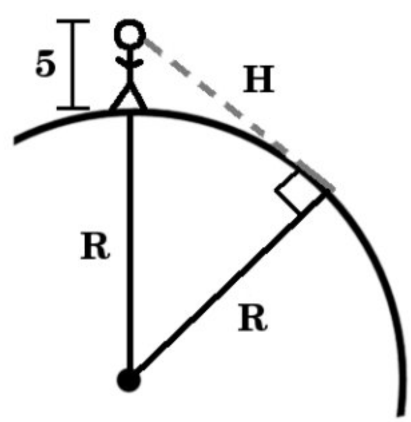
We'll warm up with a well-known (in some circles!) question: How far is the horizon?

We'll pretend things are simple. We'll pretend the Earth is a sphere, and suppose we're at the top of a tall mountain, say, 5 metres high. (Yes - I know that's not very tall really, but bear with me ...)

We can create a right-angled triangle with one corner at the centre of the Earth, one corner at our position, and one where our line-of-sight tangents the Earth's surface.

Our good friend Pythagoras now steps up and says that $R^2 + H^2 = (R + 5)^2$, which can be expanded and simplified and we get

$$H^2 \approx 10 \times R$$



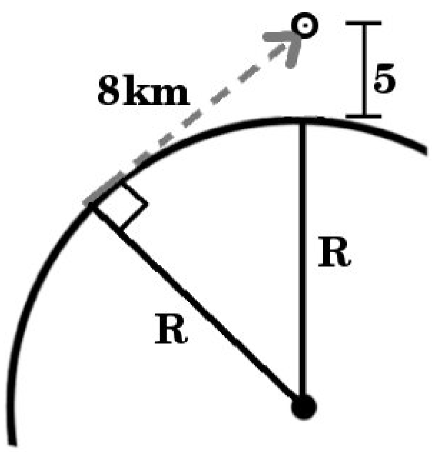
(As in this case, I'll keep the equations in the main text fairly simple throughout and expand on them in the boxes on the right side of the page. Ignore them if you just want the main ideas, or if you want the challenge of working out the details yourself.)

Now, the original definition of the metre was "One ten-millionth of the distance from the North Pole to the Equator through Paris," which means the circumference of the Earth is 40 million metres, so the radius is roughly 6.4 million metres. Substituting this we get

$$H^2 \approx 64 \times 10^6, \text{ and so } H \approx \sqrt{64 \times 10^6} = 8000\ m$$

So from a height of 5 metres, the distance to the horizon is about 8000 metres, or 8km.

$R^2 + H^2 = (R+5)^2$
$R^2 + H^2 = R^2 + 2 \times 5 \times R + 5^2$
$R^2 + H^2 = R^2 + 10 \times R + 25$
so
$H^2 = 10 \times R + 25$
Then we ignore the 25.



Now let's turn it around. Suppose we're at sea level and 8000 m from the top of a 5 m high mountain. Suppose further we fire a projectile line-of-sight at the peak, ignore air resistance, and it gets there in 1 second (unlikely, I know). In one second it will fall about 5 m, because acceleration due to gravity is about $10 m/s^2$, so by the time it gets there, it will still be at sea-level. In other words, it will be grazing the Earth's (perfectly spherical) surface.

It's in orbit.

So we've just shown that subject to all our approximations, orbital velocity at grazing altitude is 8 km/s. Quite astonishing how our good friend Pythagoras is, in some sense, "Rocket Science."

So now let's be a little more general. Instead of being exactly 5 m high, let's pick an acceleration a and an amount of time t and suppose we are $at^2/2$ high.

Our Pythagorean triangle equation now becomes

$$H^2 + R^2 = (R + at^2/2)^2$$

That seems arbitrary, I know, but distance fallen in time t under acceleration a is given by $d = at^2/2$ so we're at a height such that something will fall that distance in time t.

We simplify, divide through by t^2, throw away the irrelevant small part, and then remember that distance over time is velocity. That means we get

$$v^2 = aR$$

or equivalently,

$$a = v^2/R$$

$H^2 + R^2 = (R + at^2/2)^2$
$H^2 + R^2 = R^2 + 2 \times R \times at^2/2 + (at^2/2)^2$
Ignoring the last term (because it's small) and cancelling the R^2 term, we get:

$H^2 = R \times a \times t^2$
$(H/t)^2 = R \times a$
$v^2 = R \times a$

Suddenly we have the formula for acceleration in a circle.

Which is nice.

What does this have to do with the Moon?

You may ask why something moving in a circle is accelerating? Well, its speed may not be changing, but its direction is. If left alone its direction wouldn't change, so something must be pushing on it, changing its direction. Newton tells us that Force is Mass times Acceleration, so if there's a force, there must be acceleration.

I've also been somewhat cavalier about ignoring small quantities and so forth. In truth there are some details about limits and such like, and that's where the serious calculus should be done.

If we suppose the Moon to be moving in a circular orbit, and supposing the radius of that orbit to be M, we can now say that its acceleration in orbit is v^2/M. So if only we knew how far away it was, and its velocity, we would know its acceleration.

But if we know its distance then we do know its velocity, because we know it takes 29.53 days from full moon to full moon. Correcting for sidereal time, that means it takes 27.32 days to make a complete circuit of the Earth. Call that time P. Therefore the Moon's velocity in orbit is

$$(2\pi M)/P$$

When the Moon goes around the Earth, the Earth is also going around the Sun. In the 365.25 days it takes for a year the Moon goes around the Earth - apparently - some 365.25/29.53 times. However ...

The orbit of the Earth itself adds another complete rotation. From the point of view of the stars the Moon hasn't gone around 12.37 times, it's gone around 13.37 times, and that means that each orbit, from the point of view of the stars, takes 365.25/13.37 days, or 27.32 days.

Thus the sidereal orbital period of the Moon is 27.32 days.

So we have the formula for acceleration in a circle that needs the distance and velocity, but we now know both of those, so we can say that the Moon's acceleration is this:

$$a = (\tfrac{2\pi}{P})^2 M$$

Is <u>that</u> of any use?

Well, we know that acceleration due to gravity is what holds the Moon in orbit, so if only we knew how hard the Earth is pulling the Moon, then we would know that.

$$\begin{aligned} a &= v^2/M \\ &= [(2\pi M)/P]^2/M \\ &= (2\pi)^2 M/P^2 \\ &= (2\pi/P)^2 M \end{aligned}$$

But we do.

We know that acceleration at the Earth's surface, at distance R from the centre, is g. We also know that it falls off as an inverse square. Hence the acceleration due to gravity at any distance, say M, is given by:

$$a = g(R/M)^2$$

where R is the Earth's radius.

The "inverse square" bit means this. As you get further from something, the amount of gravitational force it exerts on you is less. Newton's law tells us exactly how much less.

A square that's three times the side-length has nine times the area. In the same way, if you go three times as far from something, the force it exerts on you will be nine times less.

So putting it all together we get:

acceleration in a circle : $a = g(\frac{2\pi}{P})^2 M$

acceleration due to gravity : $a = g(\frac{R}{M})^2$

Therefore $M^3 = g(\frac{RP}{2\pi})$

$$(\tfrac{2\pi}{P})^2 M = g(\tfrac{R}{M})^2$$

$$M^3 = g(\tfrac{RP}{2\pi})^2$$

And we know everything on the right hand side, except g.

But we can find g with a pendulum and a stopwatch. (I knew you'd be wondering where they came in.)

We know that the time taken for a complete swing of a pendulum is given by the formula:

$$T = 2\pi\sqrt{L/g}$$

Rearranging this we get:

$$g = L(\tfrac{2\pi}{T})^2$$

We can substitute that into our earlier formula and we get

$$M^3 = L(\tfrac{RP}{T})^2$$

And now we know everything!!

For very small displacements, the force pulling a pendulum back into the vertical is proportional to the amount it's been displaced. More specifically, the ratio of restoring acceleration to gravity is the same as the ratio of displacement to pendulum length. As a formula:

$$x''/g = x/L$$

That means that the formula for its motion (in simple form) can be written as

$$x = a \cdot \cos(kt)$$

where a is the amplitude of the swing, and $k = \sqrt{g/L}$

One cycle is then complete when $kt = 2\pi$ and so one complete cycle of the pendulum takes $2\pi\sqrt{L/g}$ seconds.

Of course we have to go away and construct a pendulum, and then we have to measure how long it takes to swing. Typically we measure 10 swings, both back and forth, and then divide the total time by 10. We should also do that several times to make sure we get error bars on the result, because each one will vary slightly. There's lots to do here.

So what do we get? Here are my results:

Length of the pendulum :	0.45 metres
Period of a pendulum :	1.345 seconds
Moon's orbital period :	27.32×86400 seconds
Radius of the Earth :	$40 \times 10^6/(2\pi)$ metres

A moment's work with a calculator, and we compute that the distance to the Moon is 383 thousand kilometres.

Which is the right answer.

Of course, the Moon's orbit isn't circular, the Earth isn't of constant radius, nor is it a sphere, and we've assumed that the metre is one ten millionth of the distance from the North Pole to the Equator. But even so, we're not just in the right ball park, we're smack in the middle of the true range.

Not bad for a few sums.

As a final note to comment on the fundamental inter-connectedness of things in mathematics and science, the correction for sidereal time is related to a problem from Martin Gardner's Mathematical Circus:

If you roll a coin around a fixed coin of the same size, keeping the rims together to prevent sliding, how many rotations will it make in a round trip?

A dynmical approach to necklaces and words

Cristina Serpa
Faculty of Sciences of the University of Lisbon

Abstract

We introduce concepts of formal languages, such as necklaces and Lyndon words. We give examples, some classical and others visually more attractive and playful. In parallel, in terms of dynamical systems, we define a particular circle map. Associating all the listed concepts we construct simultaneously aperiodic necklaces and periodic orbits of the defined circle map, that is, we show, by examples, that given a finite alphabet and a circle map as defined, there is a natural bijection between the set of all the aperiodic necklaces of length n over this alphabet and the set of all the n-periodic orbits of this circle map.

Introduction

We are going to present a relation between concepts of two different areas of mathematics: combinatorics on words and dynamical systems. Our aim is to give a light approach emphasizing the playful side of the subject. The author submitted a paper together with Jorge Buescu (see (Serpa & Buescu)) where a more rigorous and thorough of the matter is developed. In that paper it is possible to understand the mathematical reason of the bijection presented here. The relation between concepts is based on results from number theory that we will not develop here, but can be seen in (Serpa & Buescu).

For completeness we recall the definition of Möbius function (see (Hardy & Wright, 1979)):

Definition 1.1. Let $n = p_1 \cdots p_k$ be the prime factorization of n. The *Möbius function* $\mu(n)$ is defined as follows:

(i) $\mu(1) = 1$;

(ii) $\mu(n) = 0$ if n has a squared prime factor;

(iii) $\mu(p_1 p_2 \cdots p_k) = (-1)^k$ if all the primes $p_1, p_2, \ldots, p_k$ are different.

Necklaces and Lyndon words

In this section we introduce some concepts of formal languages and combinatorics on words defined in (Berstel & Perrin, 2007) and (Rozenberg & Salomaa, 1997) and give some illustrative examples.

Definition 2.1. An *alphabet* is a finite nonempty set of symbols.

We mention some common examples.

Example 2.2. The binary alphabet $\{0, 1\}$.

Example 2.3. The 10-digit alphabet $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Example 2.4. The English language alphabet $\{a, b, c, \ldots, x, y, z\}$ as well as other language alphabets.

Example 2.5. The alpha-numeric alphabet $\{0, 1, \ldots, 8, 9, a, b, c, \ldots, x, y, z\}$.

Example 2.6. The hexadecimal alphabet $\{0, 1, \ldots, 8, 9, a, b, c, d, e, f\}$.

We now show other alternative alphabets.

Example 2.7. The suit of cards alphabet $\{\spadesuit, \clubsuit, \heartsuit, \diamondsuit\}$.

Example 2.8. The 3-geometric forms alphabet $\{\square, \triangle, \bigcirc\}$.

Example 2.9. The 7-colors alphabet {■, ■, ■, ■, ■, ■, ■}.

Example 2.10. The zodiac alphabet which include the symbols of Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius and Pisces.

Definition 2.11. A *word* or *string* over an alphabet $\sum$ is a finite sequence of symbols taken from $\sum$. The *catenation* of two words is the word formed by juxtaposing the two words together, i.e., writing the first word immediately followed by the second word, with no space in between. A *factorization* of a word u is any sequence $u_1, \ldots, u_t$ such that $u = u_1 \ldots u_t$.

For a pair (u, v) of words we define four relations:

1. u is a *prefix* of v if there exists a word z such that $v = uz$, and $\text{pref}_k(v)$ is the prefix of v of length k;
2. u is a *suffix* of v if there exists a word z such that $v = zu$, and $\text{suf}_k(v)$ is the suffix of v of length k;
3. u is a *factor* of v if there exists words z and z' such that $v = zuz'$;
4. If $v = uz$ we write $u = vz^{-1}$ or $z = u^{-1}v$, and say that u is the *right quotient* of v by z, and that z is the *left quotient* of v by u.

Definition 2.12. Consider the cyclic permutation $c : \sum \to \sum$ defined by

$$c(u) = \text{pref}_1(u)^{-1}\, u\, \text{pref}_1(u) \tag{1}$$

for $u \in \sum$. We say that two words u and v are *conjugates* if, and only if, there exists a k such that $v = c^k(u)$.

Remark 2.13. It is easily shown that conjugacy is an equivalence relation.

As in (Berstel & Perrin, 2007) we name each equivalence class under conjugation by *necklace*.

Definition 2.14. Let $u = a_1 \dots a_n$, with $a_i \in \sum$. A period of u is an integer p such that

$$a_{p+i} = a_i \text{ for } i = 1, \dots, n-p. \tag{2}$$

The smallest p satisfying (2) is called the *period* of u, and it is denoted by $p(u)$. A word with a given period is called periodic word, otherwise is aperiodic. The words in the necklace $\left[\mathrm{pref}_{p(u)}(u)\right]$ are called *cyclic roots* of u. We say that a word $u \in \sum$ is *primitive* if it is not a proper integer power of any of its cyclic roots; its necklace is an *aperiodic necklace*. A *Lyndon word* $u \in \sum$ is a word which is primitive and the smallest one in its necklace with respect to the lexicographic ordering.

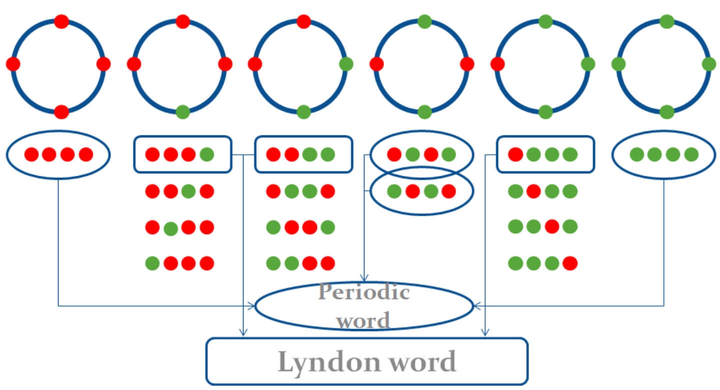
Remark 2.15. We note that primitive words are aperiodic words and an equivalent condition of primitiveness is

$$\forall z \in \sum : u = z^n \Rightarrow n = 1 \text{ (i.e., } u = z) \,. \tag{3}$$

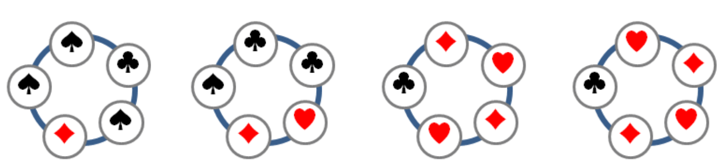
As noted in (Berstel & Perrin, 2007), considering an alphabet with a symbols, the number of aperiodic necklaces of length n is given by $\mathcal{N}_n(a)$, where

$$\mathcal{N}_n(a) = \frac{1}{n}\sum_{d|n} a^d \mu\left(\frac{n}{d}\right). \tag{4}$$

Example 2.16. We next exhibit all possible necklaces of length 4 from an alphabet of two symbols and corresponding words.



Example 2.17. Consider the suit of cards alphabet with the order ♠ ≺ ♣ ≺ ♡ ≺ ♢. The necklaces from the picture correspond, respectively, to the Lyndon words: ♠♠♣♠♢, ♠♣♣♡♢, ♣♢♡♢♡ and ♣♡♢♡♢.



Necklaces in a circle map

We briefly present the needed definitions from dynamical systems which can be found, e.g., in (Hasselblatt & Katok, 2003).

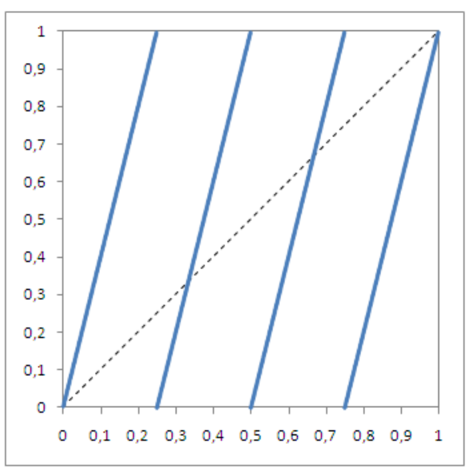
Definition 3.1. Let X be a topological space and $f : X \to X$ a continuous map. Given a point $x \in X$, the sequence $(x,\ f(x),\ f(f(x)),\ ...,\ f^n(x),\ ...)$ is called the *orbit* of x under f. A *fixed point* of f is a point such that $f(x) = x$. The *set of fixed points* of f is denoted by $Fix(f)$. A *periodic point* is a point x such that $f^n(x) = x$ for some $n \in \mathbb{N}$, that is, a point in $\mathrm{Fix}(f^n)$. Such n is said to be a *period* of x and its orbit is a *n-periodic orbit*. The smallest such n is called the *prime period* of x. The number of periodic points of f of period n is denoted by $\mathrm{Per}_n(f)$, that is, the number of fixed points for f^n.

Now we introduce a circle map, as defined in (Frame, Johnson, & Sauerberg, 2000).

Definition 3.2. For each integer $a \geq 2$ we define the circle map $g_a : S^1 \to S^1$ as

$$g_a(x) = a \cdot x \pmod 1. \tag{5}$$

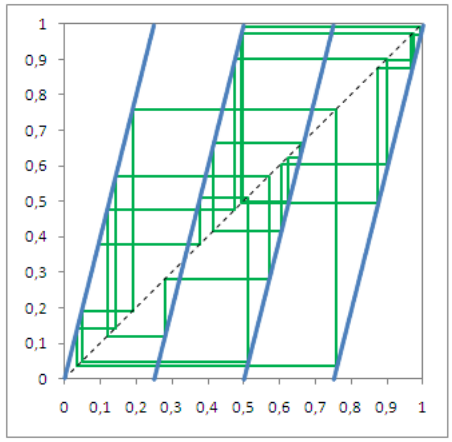
Example 3.3. Consider the graph of $g_4(x)$ shown in the following figure.



Given any initial condition in $[0, 1]$, the corresponding orbit may be constructed by graphical analysis.

Example 3.4. We exhibit the first 25 iterates of a randomly chosen initial condition. This orbit seems not to be periodic.

In fact, since g_a preserves Lebesgue measure, the probability that a randomly chosen point will have a dense (in fact, uniformly distributed) orbit on $[0, 1]$ is 1; see, e.g., (Cornfeld, Fomin, & Sinai, 1982).



We are interested in the 0-measure set of periodic orbits, since we are going to associate aperiodic necklaces to periodic orbits of a map g_a. This is possible because of the next result.

Theorem 3.5. *The number of periodic points of prime period n of g_a is*

$$N_n(a) = \sum_{d|n} a^d \mu\left(\frac{n}{d}\right). \tag{6}$$

Proof. See (Levine, 1999). □

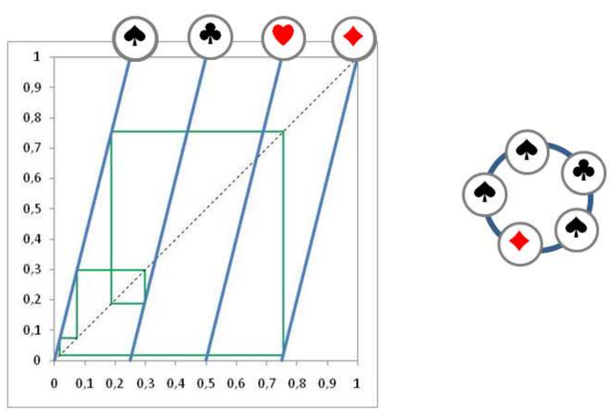
Knowing that each periodic orbit of prime period n passes through exactly n periodic points of prime period n, we easily conclude that the number of orbits of prime period n of g_a is given by $\mathcal{N}_n(a)/n$. We thus have:

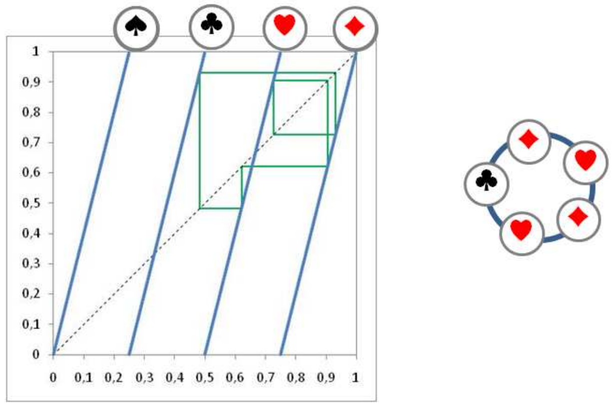
Corollary 3.6. The number of periodic orbits of prime period n of g_a is

$$\mathcal{O}_n(a) = \frac{1}{n}\sum_{d|n} a^d \mu\left(\frac{n}{d}\right). \tag{7}$$

Thus the cardinalities $\mathcal{N}_n(a)$, of the set of all aperiodic necklaces of length n over an alphabet with a symbols, and $\mathcal{O}_n(a)$ of the set of all periodic orbits of prime period n of g_a, are the same. We next show how a natural bijection between these sets may be constructed.

Example 3.7. From previous examples of necklaces we associate them to periodic orbits of g_4.





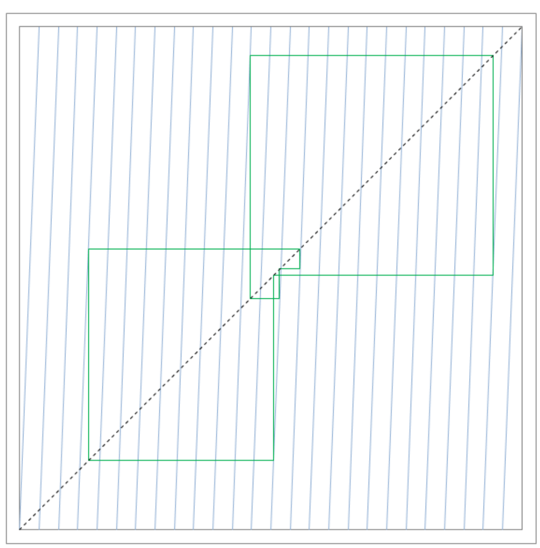
Visually, as noted in (Serpa & Buescu), this method can be identified as a loom that produces aperiodic necklaces. First each line segment should have a different color. Then just make a string go through an orbit and put a colored bead every time this touches the line segments of the circle map.

Conversely, we may derive a formula for obtaining the points of a periodic orbit of g_a from a correspondent necklace or primitive word. We can calculate the points of a periodic orbit of g_a, using the sequence of letters of a given word and its necklace (see Serpa & Buescu):

$$x = \frac{\sum_{k=1}^{m} j_k a^{m-k}}{a^m - 1}. \tag{8}$$

Considering the English alphabet, the corresponding circle map is $26 \cdot x \pmod 1$ and the set of all orbits resulting from the English words would compile an orbit English dictionary. Note that almost all English words are primitive words. It is rare to find exceptions, such as *cuscus*, *gaga*, *mama* and *yoyo*. This ones would be excluded from that dictionary. Let us see an example.

Example 3.8. The orbit associated to the word *Lyndon*.



Conclusions

Here we presented a light approach to the issue, stressing the visual aspects and omitting the proofs and developments. However, we give the reader references which allow him to understand more deeply the fundamentals of the matter. We think there are possible consequences arising for both areas (combinatorics on words and dynamical systems) yet unexplored.

Acknowledgments

The author acknowledge the organization of the Recreational Mathematics Colloquium II for giving the opportunity to present these ideas at this event and to her supervisor Jorge Buescu.

References

Berstel, Jean, & Perrin, Dominique (2007). *The origins of combinatorics on words*, European Journal of Combinatorics, 28 996-1022.

Cornfeld, I., Fomin, S., & Sinai, I. (1982). *Ergodic Theory*, Springer - Verlag.

Frame, M., Johnson, B., & Sauerberg, J. (2000). *Fixed Points and Fermat: A Dynamical Systems Approach to Number Theory*, The Mathematical Association of America, Monthly 107, May, 422-428.

Hardy, G. H., & Wright, E. M. (1979). *An Introduction to the Theory of Numbers*, Oxford University Press, Fifth Edition.

Hasselblatt, Boris, & Katok, Anatole (2003). *A First Course in Dynamics with a panorama of recent developments*, Cambridge University Press.

Levine, L. (1999). *Fermat's Little Theorem: A Proof by Function Iteration*, Mathematics Magazine, Vol. 72, No. 4, October, 308-309.

Serpa, Cristina, & Buescu, Jorge. *Circle maps and Lyndon words - a dynamical approach to aperiodic necklaces*, submitted.

Rozenberg, G., & Salomaa, A., Eds. (1997). *Handbook of Formal Languages*, Vol. 1 Word Language Grammar, Springer.

Recreational mathematics in Leonardo of Pisa's *Liber abbaci*

Keith Devlin
Stanford University

Leonardo of Pisa's classic, medieval text *Liber abbaci* was long believed to have been the major work that introduced Hindu-Arabic arithmetic into Europe and thereby gave rise to the computational, financial, and commercial revolutions that began in thirteenth century Tuscany and from there spread throughout Europe. But it was not until 2003 that a key piece of evidence was uncovered that convinced historians that such was indeed the case.

I tell that fascinating story in my book *The Man of Numbers: Fibonacci's Arithmetic Revolution* (Devlin, 2011). In this article, I will say something about Leonardo's use of what we now call recreational mathematics in order to help his readers master the arithmetical techniques his book described. One example is very well known: his famous rabbit problem, the solution to which gives rise to the Fibonacci sequence. But there are many others.

***Liber abbaci* manuscripts**

Leonardo lived from around 1170 to 1250; the exact dates are not know, nor is it known where he was born or where he died, but it does appear he spent much of his life in Pisa. As a teenager, he traveled to Bugia, in North Africa, to join his father who had moved there to represent Pisa's trading interests with the Arabic speaking world. It was there that he observed Muslim traders using what was to him a novel notation to represent numbers and a remarkable new method for performing calculations. That system had been developed in India in the first seven centuries of the first Millennium, and had been adopted and thence carried northwards by the Arabic-Speaking traders who traveled the Silk Route. He wrote *Liber abbaci* (the title translates as "Book of Calculation") in 1202, soon after he returned to Pisa.

Leonardo began this book with the words "Here begins the Book of Calculation composed by Leonardo Pisano, Family Bonacci, in the year 1202." The Latin phrase that I have translated as "family Bonacci" was *filius Bonacci*. In 1838, the historian Gillaume Libri took that phrase as the basis for a surname he gave him, "Fibonacci." (Surnames were not common in medeival times.)

No copies of the 1202 manuscript have survived, but much later in Leonardo's life, in 1228, by which time he had become famous throughout Italy, he wrote a much enlarged, second edition, of which fourteen copies exist today in various degrees of completeness. Seven are mere fragments, consisting of between one-and-a-half and three of the book's fifteen chapters. Of the remaining seven, more substantial manuscripts, three are complete or almost so, and are generally regarded as the most significant. Those three are all in Italy.

One is housed in the Biblioteca Communale di Siena (Siena Public Library), where it has the reference number L.IV.20. It is generally believed to date from the thirteenth century, and according to some scholars is possibly the oldest, with a possible date of 1275, perhaps not long after Leonardo died – though others have suggested it may have been written as much as a century later.

Another, also believed to date from the late thirteenth, or perhaps the early fourteenth century, is in the Biblioteca Nazionale Centrale di Firenze (BNCF – Florence National Central Library), where it is listed in the catalogue as Conventi Sopressi C.1.2616. This manuscript is complete, which probably explains why the publisher Baldassarre Boncompagni used it as the basis for the first printed edition, which he brought out in the mid nineteenth century, even though it is not the best preserved, and perhaps not the oldest.

The third is in the Vatican Library in Rome, where it bears the reference mark Vatican Palatino ♯1343. This manuscript, from which Chapter 10 is missing, is also believed to date back to the late thirteenth century.

Of the remaining, more fragmented manuscripts, four are housed in the BNCF, along with the one mentioned above, one is in the Biblioteca Laurentiana Gadd in Florence (Gadd. Reliqui 36, dated to the 14^{th} century), one in the Biblioteca Riccardiana in Florence, one in the Biblioteca Ambrosiana in Milan, one in the Biblioteca Nazionale Centrale in Naples, and three in Paris (one in the Bibliothèque Mazarine, two in the Bibliothèque National de France).

Contents of *Liber abbaci*

Liber abbaci was a huge work. For instance, the Siena manuscript has 448 sides. The first, and only, printed edition of Leonardo's Latin text, published in Rome in 1857 by Baron Baldassare Boncompagni, an Italian bibliophile and medieval mathematical historian, has 387 densely packed pages.[1] The printed English language translation of *Liber abbaci*, by Laurence Sigler, which is based on Boncompagni's edition and published in 2002, runs to 672 pages (Sigler, 2002). It is the only translation of Leonardo's text into a modern language.

Following a dedication and a short prologue, Leonardo divided *Liber abbaci* into fifteen chapters. Their titles vary from manuscript to manuscript, suggesting that the scribes who made copies felt free to make what they felt were clarifying improvements.

Chapter 1. On the recognition of the nine Indian figures and how all numbers are written with them; and how the numbers must be held in the hands, and on the introduction to calculations

Chapter 1 occupies 6 pages in Sigler's translation, and describes how to write – and read – whole numbers in the Hindus' decimal system. For large numbers, the numerals are grouped in threes to facilitate reading.

[1] *Liber abbaci* was the first volume of a two-volume, printed collection of all of Leonardo's works that Boncompagni published in Rome under the title *Scritti di Leonardo Pisano*. The second volume, containing all of Leonardo's other works, appeared in 1862.

Chapter 2. On the multiplication of whole numbers

This is a 16-pages-long "how to" manual.[2] The approach differs little from the one used today to teach children how to multiply two whole numbers together. Leonardo begins with the multiplication of pairs of two-digit numbers and of multi-digit numbers by a one-place number, and then works up to more complicated examples.

Chapter 3. On the addition of them, one to the other

Chapter 3 is short, with just 5 pages of instructions. Giving a hit of things to come, he describes a procedure for keeping expenses in a table with columns for librae, soldi, and denari, the coinage used at the time (pounds, shillings, and pence in old English terms).

Chapter 4. On the subtraction of lesser numbers from greater numbers

This, the shortest chapter in the book, occupies a mere 3 pages of Sigler's translation. The title explains the contents.

Chapter 5. On the divisions of integral numbers

Chapter 5 focuses on divisions by small numbers and on simple fractions. It has 28 pages of instructions. It describes the familiar "long-division algorithm," still taught today in many schools.

Chapter 6. On the multiplication of integral numbers with fractions

The topic is what are today called mixed numbers, numbers that comprise both a whole number and a fractional part. Leonardo explains that you calculate with them by first changing them to fractional form (what we would today call "improper fractions"), computing with them, and then converting the answer back to mixed form. This chapter takes up 22 pages.

Chapter 7. On the addition and subtraction and division of numbers with fractions and the reduction of several parts to a single part

Leonardo fills 28 pages showing how to combine everything that has been learned so far.

Chapter 8. On finding the value of merchandise by the Principal Method

Chapter 8 provides the first real examples of practical mathematics, in the form of 51 pages of worked examples on the value of merchandise, using what we would today call reasoning by proportions. For example, Leonardo asks: if 2 pounds of barley cost 5 soldi, how much do 7 pounds cost? He then proceeds to show how to work out the answer. He makes use of simple diagrams of proportion, which he calls the "method of negotiation." Examples include monetary exchange, the sale of goods by weight, and the sale of cloth, pepper, cheese, canes, and bales.

Chapter 9. On the barter of merchandise and similar things

Leonardo presents another 33 pages (in the Sigler translation) of worked practical examples extending the discussion from the previous chapter. There are problems on the barter of common things, on the sale and purchase of money, on horses that eat barley in a certain number of days, on men who plant trees, and men who eat corn.

[2] The page counts are for Sigler's English language translation.

Chapter 10. On companies and their members

These 14 pages provide worked examples on investments and profits of companies and their members, showing how to decide who should be paid what.

Chapter 11. On the alloying of monies

Chapter 11 occupies 31 pages in Sigler's English language translation. The need for the methods Leonardo describes in this chapter was considerable. At that time, Italy had the highest concentration of different currencies in the world, with 28 different cites issuing coins during the course of the Middle Ages, seven in Tuscany alone. Their relative value and the metallic composition of their coinage varied considerably, both from one city to the next and over time. This state of affairs not only meant good business for money changers – and *Liber abbaci* provided plenty of examples on problems of that nature – but with governments regularly re-valuing their currencies, gold and silver coins provided a more stable base, and since most silver coinage of the time were alloyed with copper, problems of minting and alloying of money were important.

Chapter 12. On the solutions to many posed problems

This enormous chapter filled a staggering 186 pages with miscellaneous worked examples. Its primary focus is algebra. Not the symbolic reasoning we associate with the word today, rather "algebraic reasoning," expressed in ordinary language (and often referred to as "rhetorical algebra"). Much of Leonardo's focus is on applications of what is generally known as the "method of false position," which he refers to as the "tree method." This is a procedure used to solve problems equivalent (in modern terms) to a simple linear equation of the type $Ax = B$. The solver first picks an approximate answer and then reasons to adjust it to give the correct solution. Many of the problems Leonardo looks at he called "tree problems," which is why he speaks of solving them by the "method of trees." This is a class of problems he named after a particular puzzle he introduces in the chapter, where you want to know the total length of a tree when you are given the proportion that lies beneath the ground.

He also shows how to solve the same kinds of problems using what he called the "direct method" (*regula recta*), where you begin by calling the sought-after quantity a "thing" (*res*), and then forming an equation (expressed in words), which is then solved step-by-step to give the answer. Expressed symbolically, this is precisely the modern algebraic method. It was known to the Arab scholars, and was described by the Persian mathematician al-Khwārizmī around 830 in the book from whose title the modern word "algebra" stems.

Many of the problems Leonardo presents are of a financial nature, providing the businessman of the thirteenth century and subsequent with some extremely powerful tools that helped to revolutionize European trade and commerce.

Chapter 13. On the method elchataym and how with it nearly all problems of mathematics are solved

In modern terminology, *elchataym* is a rule, known also as "double false position," used to solve one or more linear equations. The word "elchataym" is Leonardo's Latin transliteration of the Arabic *al-khata'ayn*, which means "the two errors". The name reflects the fact that you start with two approximations to the sought-after answer, one too low, the other too high, and then reason to adjust both until the correct answer is arrived at. It can be used to solve linear equations not only of the form $Ax = B$, for which single false position can be used, but also the more general form $Ax + B = C$. This chapter provides 41 pages of worked examples. Leonardo formulates the problems in several ingenious ways, in terms of snakes, four-legged animals, eggs, business ventures, ships, vats full of liquid which empty through holes, how a group of men should share out the proceeds when they find a purse or purses, subject to various conditions, how a group of men should each contribute to the cost of buying a horse, again under various conditions, as well as some in pure number terms.

Chapter 14. On finding square and cubic roots, and on the multiplication, division, and subtraction of them, and on the treatment of binomials and apotomes and their roots

Leonardo's penultimate chapter offers 42 pages of worked examples. His main focus is on methods for handling roots. He uses the classifications given by Euclid in Book X of *Elements* for the sums and differences of unlike roots, namely binomials and apotomones. (The discovery that $\sqrt{2}$ is irrational led the ancient Greeks to a study of what they called "incommensurable magnitudes." Euclid's term for a sum of two incommensurables, such as $\sqrt{2} + 1$, was *binomial* (a "two-name" magnitude), and a difference, such as $\sqrt{2} - 1$, he called an *apotome*. Handling incommensurables by means of what we would now regard as algebraic expressions was a common feature of Greek and medieval mathematics.)

In terms of mathematical content, Chapter 14 is little more than a collection of known methods and results, and Leonardo presents nothing significant not already found in *Elements*.

Chapter 15. On pertinent geometric rules and on problems of algebra and al-muchabala

This final chapter occupies 85 pages, again filled with worked examples. In modern terms, al-muchabala corresponds to manipulating the two sides of an equation while keeping it balanced. Once you know that, the chapter title says it all. Leonardo's approach differs little from that found in al-Khwārizmī's earlier book on algebra.

Word problems and recreational mathematics in *Liber abbaci*

It is with Chapters 8 and 9 that the reader first encounters real-world examples. Many of these involve items called "Pisan rolls"; the Pisan roll was a unit of weight, equal to 12 Pisan ounces.

Units of weight differed from one city to another. One worked problem in Chapter 8 is titled:[3]

On finding the worth of Florentine rolls when the worth of those of Genoa is known. [p.148]

A typical worked problem in this section of the book starts like this:

If one hundredweight of linen or some other merchandise is sold near Syria or Alexandria for 4 Saracen bezants, and you will wish to know how much 37 rolls are worth, then ... [p.142]

[3]Page numbers refer to (Sigler, 2002).

Chapter 10, on companies and their members, demonstrates obviously valuable methods for solving problems such as determining the payouts in the following scenario:

Three men made a company in which the first man put 17 pounds, the second 29 pounds, the third 42 pounds, and the profit was 100 pounds. [p.220]

Toward the end of Chapter 11, we find a curious problem that became quite well known to mathematicians (though nothing like as famous as his "rabbit problem"). It is called "Fibonacci's Problem of the Birds". Here is what Leonardo asks: [p.256]

On a Man Who Buys Thirty Birds of Three Kinds for 30 Denari

A certain man buys 30 birds which are partridges, pigeons, and sparrows, for 30 denari. A partridge he buys for 3 denari, a pigeon for 2 denari, and 2 sparrows for 1 denaro, namely 1 sparrow for 1/2 denaro. It is sought how many birds he buys of each kind.

What makes this problem particularly intriguing is that, on the face of it you don't have enough information to solve it. You arrive at that conclusion as soon as you try to solve it using modern symbolic algebra. If you let x be the number of partridges, y the number of pigeons, and z the number of sparrows, then the information you are given leads to two equations:

$x + y + z = 30$ (the number of birds bought equals 30)

$3x + 2y + \frac{1}{2}z = 30$ (the total price paid equals 30)

But as everyone learns in the high-school algebra class, you need three equations to find three unknowns. Well, in general that is true, but in this case you have one crucial additional piece of information that enables you to solve the problem. I'll give the solution to the problem at the end of the article. (Leonardo, as usual, presents the solution in words, not symbols, but apart from that, the solution I will give is his.)

Some of Leonardo's problems are presented in more whimsical terms. For instance, many are like this one in Chapter 12:

A certain lion is in a certain pit, the depth of which is 50 palms, and he ascends daily 1/7 of a palm, and descends 1/9. It is sought in how many days will he leave the pit. [p.273]

But for the most part, the examples of *Liber abbaci* are couched in self-evidently practical terms. In fact, Chapter 12 is a mammoth piece of work that presents 259 worked examples, each of which Leonardo works through in great detail. Some examples require only a few lines to solve, others spread over several densely packed pages. In Sigler's English language translation, the entire chapter takes up 187 printed pages.

Like mathematics teachers and authors before him and since, Leonardo clearly knew that many of the people who sought to learn from him would have little interest for theoretical, abstract problems. And so, in order to explain how to use the new methods he learned during his visit to North Africa, Leonardo looked for ways to dress up the abstract ideas in familiar, everyday clothing. The result is that many of the problems he gave, while given with obvious practical applications in mind, would be classified today as "recreational mathematics."

For instance, he presents a series of "purse problems" to try to put into everyday terms the mathematical problem associated with dividing up an amount of money – or anything else that people may want to divide up – according to certain rules. The first one goes like this:

Two men who had denari found a purse with denari in it; thus found, the first man said to the second, If I take these denari of the purse, then with the denari I have I shall have three times as many as you have. Alternately the other man responded, And if I shall have the denari of the purse with my denari, then I shall have four times as many denari as you have. It is sought how many denari each has, and how many denari they found in the purse. [p.317]

Students today would be expected to solve this problem using elementary algebra (equations), and it takes at most a few lines. But modern symbolic algebra is a much later invention. Leonardo filled almost half a parchment page with his solution. (At heart, it is the same solution as today's algebra student will – or at least should – come up with, but without the simplicity given by symbolic equations it takes a lot more effort, and a great deal more space on the page, to work through to the answer.)

More complicated variations follow, including a purse found by three men, a purse found by four men, and finally a purse found by five men. Each problem took a full parchment sheet to solve. Adding still more complexity, he presented a particularly challenging problem in which four men with denari find four purses of denari, the solution of which fills four entire pages of Sigler's English language translation. In all, Leonardo presented eighteen different purse problems, which occupy nineteen-and-a-half pages of the English language translation.

Although many of the variants to the purse problem that Leonardo presents seem to be of his own devising, the original problem predated him by at least four hundred years. In his book *Ganita Sara Sangraha*, the ninth-century Jain mathematician Mahavira (ca. 800 – 870) presented his readers with this problem:

Three merchants find a purse lying in the road. The first asserts that the discovery would make him twice as wealthy as the other two combined. The second claims his wealth would triple if he kept the purse, and the third claims his wealth would increase five fold.

The reader has to determine how much each merchant has and how much is in the purse. This is precisely Leonardo's first purse problem in Chapter 12 of *Liber abbaci*. Presumably Leonardo came across the puzzle by way of an Arab text.

Leonardo's purse problems involve divisions that require only whole numbers. To explain how to proceed when fractions are involved, he used a different scenario his readers could relate to: buying horses. On page 337 of Sigler's translation, we read:

Here Begins the Fifth Part on the Purchase of Horses among Partners According to Some Given Proportion.

The first horse problem reads:

Two men having bezants found a horse for sale; as they wished to buy him, the first said to the second, If you will give me 1/3 of your bezants, then I shall have the price of the horse. And the other man proposed to have similarly the price of the horse if he takes 1/4 of the first's bezants. The price of the horse and the bezants of each man are sought. [p.337]

Again, a mathematics student today would solve this problem using (symbolic) algebraic equations. Leonardo solved the problem using arithmetic:

You put 1/4 1/3 in order, and you subtract the 1 which is over the 3 from the 3 itself; there remains 2 that you multiply by the 4; there will be 8 bezants, and the first has this many. Also the 1 which is over the 4 is subtracted from the 4; there remains 3 that you multiply by the 3; there remains 9 bezants, and the other man has this many. Again you multiply the 3 by the 4; there will be 12 from which you take the 1 that comes out of the multiplication of the 1 which is over the 3 by the 1 which is over the 4; there remain 11 bezants for the price of the horse; this method proceeds from the rule of proportion, namely from the finding of the proportion of the bezants of one man to the bezants of the other; the proportion is found thus. [p.337]

This definitely does not read like a modern day arithmetic textbook. What Leonardo is doing is explain, step-by-step, what digits you must write where, and what you do to them. A modern textbook would supply an algebraic formula into which you can simply plug in numbers, but algebraic notation was still several centuries away. Instead, Leonardo had to convey the method by giving many concrete examples, each with its unique twist and each using slightly different numbers.

Thirty-six pages and twenty-nine horse-type problems later, Leonardo evidently decided he had provided enough variations that his readers will have mastered the general technique. Along the way, he worked out a problem in which five men buy five horses [p.350], another, particularly tricky puzzle that he titles

A Problem Proposed to Us by a Most Earned Master of a Constantinople Mosque [p.362]

in which five men buy not a horse but a ship, and another problem where seven men buy a horse, which, despite its seemingly greater complexity, turns out to be less intricate to solve [p.366].

With many of his fellow citizens frequent travelers, Leonardo knew that money problems about traveling were sure to arouse wide interest, so these provide his next set of examples. For his first traveler problem, he wrote:

A certain man proceeding to Lucca on business to make a profit doubled his money, and he spent there 12 denari. He then left and went through Florence; he there doubled his money, and he spent 12 denari. Then he returned to Pisa, doubled his money, and spent 12 denari, and it is proposed that he had nothing left. It is sought how much he had at the beginning. [p.372]

While the ending of his little scenario might strike a familiar chord to many a vacation traveler to Tuscany today, this particular problem has a relatively easy solution. So too do some, though not all, of the many variants of the problem that Leonardo solves in the ensuing pages. He also illustrates the same arithmetical principles and solution methods with some other problems, including several about calculating interest on house purchases. [pp.384–392].

One problem leads to the particularly nasty answer that a certain businessman walks away from a partnership in Constantinople with a profit of

$$\frac{1}{2}\,\frac{7}{8}\,\frac{1}{8}\,\frac{4}{8}\,\frac{0}{8}\,\frac{21169}{24767}\,206 \text{ bezants}$$

To read this, you need to understand that, when Europeans in Leonardo's time learned the Hindu-Arabic number system, they wrote fractions before the whole number part, built up from right to left, with each new fraction representing that part of what is to the right. For example,

$$\frac{1}{2}\,\frac{2}{3}\,\frac{4}{5} \text{ means } \frac{1}{2\times 3\times 5}+\frac{2}{3\times 5}+\frac{4}{5}, \text{ i.e. } \frac{29}{30}$$

The right-to-left ordering may simply be a carry-over from the writing of Arabic, although it is of interest to note that, for the most part, Arabic texts expressed Hindu-Arabic numbers rhetorically, using words instead of symbols. Leonardo would have articulated the above fraction as the Arabic mathematicians would both write and speak it: "For fifths, and two thirds of a fifth, and one half of a third of a fifth."

Decimal expansions are a special case of this notation when the denominators are all 10. For example, Leonardo would have written today's decimal number 3.14159 as

$$\frac{9}{10}\,\frac{5}{10}\,\frac{1}{10}\,\frac{4}{10}\,\frac{1}{10}\,3$$

Though decimal representation seems far simpler to us today, there was little need for it in Leonardo's time, as no one counted anything special in tenths. In fact, the method used to represent fractions was particularly well-suited for calculations involving money. In the monetary system used in medieval Pisa, 12 denari equaled 1 soldus and 20 soldi equaled 1 libra, so 2 librae, 7 soldi, and 3 denari would be written

$$\frac{3}{12}\,\frac{7}{20}\,2$$

Units of weight and measure could be even more complex. According to Leonardo, Pisan hundredweights (Sigler, 2002, p.128):

... have in themselves one hundred parts each of which is called a roll, and each roll contains 12 ounces, and each of which weighs 1/2 39 pennyweights; and each pennyweight contains 6 carobs and a carob is 4 grains of corn.

Imagine having to calculate with those units.

Interestingly, an Arabic arithmetic text written by al-Uqlidisi in Damascus in 952 did in fact use place-value decimals to the right of a decimal point, but no one saw any particular reason to adopt it, and so the idea died, not to reappear again for five hundred years, when Arab scholars picked up the idea once more. Decimal fractions were not used in Europe until the sixteenth century.

Fractions written after the whole number part in Leonardo's time denote multiplication. For example, 1/2 of 3.14159 could be written

$$\frac{9}{10}\,\frac{5}{10}\,\frac{1}{10}\,\frac{4}{10}\,\frac{1}{10}\,3\,\frac{1}{2}$$

The Fibonacci sequence

Leonardo's most well known connection to present day recreational mathematics comes toward the latter part of his Chapter 12. Nestled between problems involving the division of food and money, Leonardo throws in a whimsical problem about a growing rabbit population. He did not invent the problem; it dates back at least to the Indian mathematicians in the early centuries of the Current Era who developed the number system *Liber abbaci* describes. He clearly realized, however, as did his Hindu predecessors, that it is an excellent, easy problem for practicing how to use the new number system. And so he included it. What he obviously could not foresee was that, although later generations of historians of mathematics would consider *Liber abbaci* one of the most influential books of all times, it would be in associaion with that one little problem that he would be most widely known.

In what was to become his most famous passage, Leonardo wrote his way into twentieth century popular culture with these words [p.404]:

How Many Pairs of Rabbits Are Created by One Pair in One Year.

A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.

As usual, Leonardo explained the solution in full detail, but the modern reader can rapidly discern the solution method by glancing at the table Leonardo also presented, giving the rabbit population each month:

Beginning	*1*
first	*2*
second	*3*
third	*5*
fourth	*8*
fifth	*13*
sixth	*21*
seventh	*34*
eighth	*55*
ninth	*89*
tenth	*144*
eleventh	*233*
twelfth	*377*

The general rule is that each successive number is the result of adding together the previous two; thus, $1 + 2 = 3$, $2 + 3 = 5$, $3 + 5 = 8$, etc. As Leonardo observes at the end of his solution, although he has calculated the population at the end of one year, namely 377, this simple rule gives you the population after any number of months.

The numbers generated by the addition process Leonardo described to solve the rabbit problem are known today as the Fibonacci numbers. They first appeared, it seems, in the *Chandahshastra* (The Art of Prosody) written by the Sanskrit grammarian Pingala some time between 450 and 200 BCE. Prosody was important in ancient Indian ritual. In the sixth century, the Indian mathematician Virahanka showed how the sequence arises in the analysis of metres with long and short syllables. Subsequently, the Jain philosopher Hemachandra (c.1150) composed a text on them.

The Fibonacci numbers were given their name by the French mathematician Edouard Lucas in the 1870s, after the French historian Guillaume Libri gave Leonardo the nickname Fibonacci in 1838. A lot of the initial – and subsequent – fascination with them is due to the surprising frequency with which they seem to arise when you go out into the garden and count things. Many of those occurrences arise because of a well known rconnection to the Golden Ratio. But that is a well known story, repeated often – though in many cases with false claims alongside the valid ones, perhaps the most frequent false claim being that the Fibonacci sequence arises naturally in the shell of the Chambered Nautilus.[4]

APPENDIX: The solution to Leonardo's Problem of the Birds

The problem of the birds says:

A certain man buys 30 birds which are partridges, pigeons, and sparrows, for 30 denari. A partridge he buys for 3 denari, a pigeon for 2 denari, and 2 sparrows for 1 denaro, namely 1 sparrow for 1/2 denaro. It is sought how many birds he buys of each kind.

Here is a modern solution. You let x be the number of partridges, y the number of pigeons, and z the number of sparrows. The information you are given then leads to two equations:

$x + y + z = 30$ (the number of birds bought equals 30)

$3x + 2y + \frac{1}{2}z = 30$ (the total price paid equals 30)

This looks like an impossible task, since you have three unknowns but only two equations. But the problem provides you with a crucial additional piece of information that enables you to solve it: The values of the three unknowns must all be positive whole numbers. (You are told he buys three kinds of birds, so none of the unknowns can be zero, and he surely does not buy fractions of birds.)

Start by doubling every term in the second equation to get rid of that fraction:

$x + y + z = 30$

$6x + 4y + z = 60$

[4]The Chambered Nautlus shell has the shape of a logarithmic spiral, but its angle of rotation is not the goden ratio, so you don't get the Fibonacci numbers.

Subtract the first equation from the second to eliminate z:

$5x + 3y = 30$

Notice that 5 divides the first term and the third, so it must also divide y. So y is one of 5, 10, 15, etc. But y cannot be 10 or anything bigger, since then it could not satisfy that last equation! Thus $y = 5$. It follows that $x = 3$ and $z = 22$. Neat, eh?

References

Devlin, Keith (2011). *The Man of Numbers: Fibonacci's Arithmetic Revolution*, Walker Books.

Sigler, Laurence (2002). *Fibonacci's Liber abaci*, Springer-Verlag.

Syzygies played by elementary school students

Dores Ferreira
University of Minho
doresferreira@gmail.com

Pedro Palhares
University of Minho
palhares@ie.uminho.pt

Jorge Nuno Silva *
University of Minho
jnsilva@cal.berkeley.edu

Abstract

The mathematician Charles Lutwidge Dodgson, well known by the pseudonym of Lewis Carroll, besides being the famous author of "Alice's adventures in wonderland" he was also the inventor of a variety of games and puzzles. Lewis Carroll liked to play with words and one of his inventions was a word-puzzle that he named Syzygies. As part of a research involving games, in the last few months elementary school students have played Syzygies. In this paper we will present the results of this practice and its connections with mathematics education.

Introduction

Probably, men have always played games. Actually, according to Huizinga (2003), "play is older than culture" and concerning board games, men have played for more than four thousand years (Murray, 1952). Nowadays, people of all ages still like to play games and young students are perhaps the most enthusiastic. This motivating characteristic of games has been used in mathematics education as a facilitator of the teaching and learning process. Some guidelines of the Portuguese curriculum point to the use of games that may promote the development of math skills (DEB, 2001), stating games as one of the different type of mathematical experiences that students must be involved (DGIDC, 2007). This documents claim also the development of the ability to identify and explore patterns in mathematical or non mathematical contexts (DEB, 2001; DGIDC, 2007). In addition, the ability to identify patterns is referred in literature as an inherent capacity for mathematical activities (Devlin, 1997), a major ally of mathematics education (Steen, 1990) and one of the most powerful strategies for problem solving (Posamentier & Krulik, 1998).

Research involving mathematical games

The interest in games and its connection with elementary mathematics educations has been the starting point to an ongoing study involving elementary school students. The main goal of this study concerns the verification of possible relationships between games and patterns. The ability to identify patterns has been measured by a test constructed and validated for the study. This test

*Partially supported by Project PTDC/HCT/70823/2006

has 24 questions involving numeric and geometric patterns with a structure based on the structure of similar questions used by other authors, such as Krutetskii (1976). It is also based on the conclusions of Krutetskii's research, stating the existence of three types of mathematical ability: analytical, geometric and harmonic (combining the other two). The methodology is quantitative and the statistical analysis has been done using the software SPSS for Windows and the appropriate statistical tests for each condition. Until now, one of the most frequently used tests is the test of correlation. To interpret the correlation coefficient we have followed the criteria pointed by author such as Cohen & Manion (1989) or Fraenkel and Wallen (1990). Therefore, coefficients of correlation between 0,2 and 0,35 are interpreted as revealing a small relationship between variables, too small to make predictions; coefficients of correlation between 0,35 and 0,65 may have theoretical and practical importance depending on the context but they allow for group predictions; coefficients greater than 0,65 reveals a very good relationship between variables.

In a previous study, chess was the first game analysed with a sample constituted by 437 students from 3$^{\text{rd}}$ to 6$^{\text{th}}$ grades. The analysis of collected data showed the existence of a relationship between the ability to play chess and the ability to identify patterns ($r =,46; p <,01$). In the analysis it was also found that School grade affects the relationship between strength of play and the ability to identify patterns. However when we excluded its effects, still the relationship was above 0,38 (Ferreira & Palhares, 2008). The results obtained in this analysis encourage us to carry on the research taking into account others games.

Portugal has a National Championship of Mathematical Games since 2004, involving students from 1$^{\text{st}}$ to 12$^{\text{th}}$ grades and six mathematical games (Neto & Silva, 2004), three games for each level of the Portuguese school system. In 2007, the National Championship of Mathematical Games was the background of another collected data from student's participants in the final of this championship. We were curious to find if there was a relationship between the ability to find patterns and the ability to play mathematical games other than chess, as for example Traffic lights. The analysis of collected data revealed that in 3$^{\text{rd}}$ and 4$^{\text{th}}$ grades, there's a relationship between the ability to identify patterns and the ability to play the game Traffic lights ($r = -,76; p <,01$). The negative coefficient is explained by an opposite direction between variables (e.g. in the ranking of players, the highest the number, the worse the player)(Ferreira, Palhares & Silva, 2008). As the number of participants in the final of each game is small, the next step was organize championships to analyse a greater number of students. We selected three mathematical games (Traffic Lights, Dots and boxes, Wari) and we organized championships with 41 students from 3$^{\text{rd}}$ and 4$^{\text{th}}$ grades (8 to 9 years old) of a primary school. The analysis of collected data from these students revealed that in 3$^{\text{rd}}$ and 4$^{\text{th}}$ grades students, there's a relationship between the ability to identify patterns and the ability to play "traffic lights" but small ($r = -,32; P <,05$) and there's a moderate relationship between the ability to identify patterns and the ability to play "Wari" ($r = -,40; p <,01$). The discrepancy of results, between the previous analysis and thus one for made us conjecture that maybe the relationship was valid only for good players. So we decided to restrict the statistic analysis to the best players and we have obtained the following results:

- Concerning the best players of 3$^{\text{rd}}$ and 4$^{\text{th}}$ grades students there's a very strong relationship between the ability to identify patterns and the ability to play "traffic lights" ($r = -,81$; $p <\ ,05$);
- Concerning the best players of 3$^{\text{rd}}$ and 4$^{\text{th}}$ grades students there's a very strong relationship between the ability to identify patterns and the ability to play "Dots and boxes" ($r = -,79; p <,05$) (Ferreira, Palhares & Silva, 2010).

In a different analysis, Ferreira and Palhares (2009) carried out factor analysis existent on data and identified the existence of seven factors behind the ability of pattern recognition. In this analysis, they considered the data collected from more than six hundred elementary school students. The interpretation of the seven factors was as follows:

Factor 1 - numeric progressions
Factor 2 - three terms repetition: ABC ABC
Factor 3 - both geometric and numeric progressions
Factor 4 - counting
Factor 5 - odd and even numbers
Factor 6 - rotation
Factor 7 - more than one rule

The results of this analysis suggest that the ability to identify patterns may involve more specific skills such as counting, repeating, among others.

Subsequently, data collected from three primary schools and one hundred and forty-eight students from 3^{rd} and 4^{th} grades was used to analyse the relationship between the ability to identify patterns and the ability to play Traffic lights, Dots and Boxes and Wari, concerning the group of better players of each game. In this analysis, we used the Kendall's Tau coefficient (τ) because is recommended when there are too many ties (Field 2000). The analysis revealed the following results:

- Concerning better players, there's a relationship between Factor 1 and the ability to play Dots and Boxes ($\tau = -{,}45; p <{,}05$);
- Concerning better players, there's a relationship between Factor 2 and the ability to play Traffic Lights ($\tau = -{,}55; p <{,}05$);
- Concerning better players, there's a relationship between Factor 4 and the ability to play Dots and Boxes ($\tau = -{,}45; p <{,}05$);
- Concerning better players, there's a relationship between Factor 7 and the ability to play Wari ($\tau = -{,}54; p <{,}01$).

These results show us the existence of different connections between each game and a specific group of patterns. Although all these games are strategic games without hidden information, they are connected to the ability to identify patterns in a distinct way. The reason of these differences could be explained by intrinsic characteristic of each game.

Research involving Syzygies

So far, the analyses have considered only mathematical games. However, we are also concerned about the possible connections with patterns and other type of games. One of the games selected in this research was Sysygies, a word puzzle invented by Lewis Carrol in 1879 (Wakeling, 1995). Although Syzygies was classified as a puzzle by Lewis Carrol, players are in competition with each other despite having an individual performance. The classifications of games include games played by one single player as refer Murray (1952) and the definition of game given by Whitehill (2009, p. 55) define a games as "a form of play, in which players compete, each trying to emerge winner according to specific set of rules and a predeterminated end". In this research, Sysygies were played by a class of 18 students competing to have the best score in their Syzygy-problem accordingly to the rules defined by Lewis Carrol. So, in this article we intend to refer to Syzygies as a game.

The main goal of this game is to link two words following a set of definitions and rules described by Wakeling (1995). These starting words are given in a Syzygy-problem with the characteristic sense of humour of Lewis Carrol, as for example "Introduce 'Walrus' to 'Carpenter' " or "Lay 'knife' by 'fork' ". The connection between the two end-words is made by other words called links, and the letter or group of consecutive letters shared by two successive words and placed in a parenthesis between them is called a syzygy. The set of end-words, links and syzygies is a chain. A chain must have at least two links. Every letter in a Chain, which is not linked to some other, is a waste.

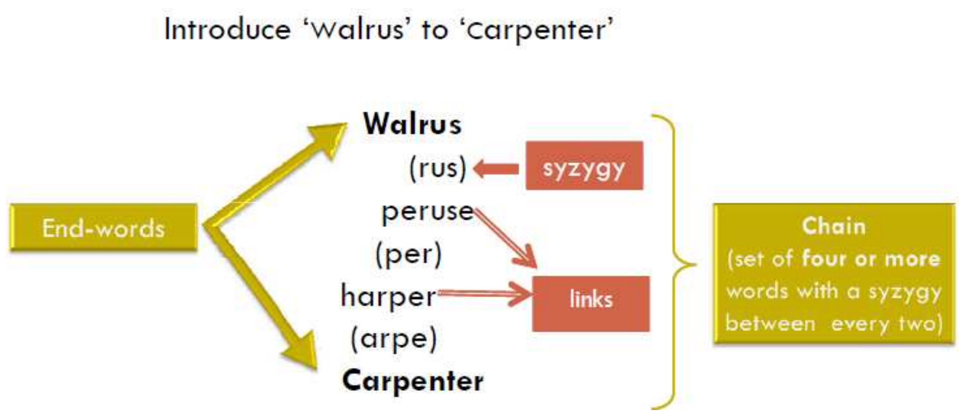


Figure 1: Example of a Syzygy-problem

Syzygies have not the same popularity as Doublets (another word game invented by Lewis Carroll) and one possible reason is the complexity of the scoring rules. However, according to Gardner (1996, p. 144), "Carroll taught the game to many child-friends, and he mentions it in his letters". The following Syzygy was found in an letter to Beatrice Earle:

Beatrice
(eatric)
theatricals
(ical)
medical
(dica)
handicapped
(appe)
appear
(pear)
pearl
(earl)
Earle

To score the game, Lewis Carrol presents a method that consist in calculate seven numbers as following:

(1) The greater number of letters in an end-Syzygy, plus the least;
(2) The least number of letters in a Syzygy;
(3) The sum of (1) plus the product of the two numbers next above (2);
(4) The number of links;
(5) The number of waste letters;
(6) The sum of twice (4) plus (5);
(7) The remainder left after deducting (6) from (3). If (6) be greater than (3), the remainder is written as 0.

In this method, the last number is the score of the chain.

To make a chain there are also some rules and some restrictions. For example, when two words begin (or end) with the same set of one or more consecutive letters, or would do if certain suffixes (or prefixes) were removed, each letter in the one set is barred with regard to the corresponding letter in the other set.

dog	onion	*in*doors	sink*ing*
door	moon	*un*done	link*s*

Figure 2: Restrictions in a Syzygy

The red letters in figure 2 are the forbidden letters and the italic highlights the prefixes or suffixes. More information about this game can be found in the book "Rediscovered Lewis Carroll Puzzles".

In our research, the sample was constituted by 18 students from 4^{th} grade (9 years old). To collect data information we used the test that measures the ability to identify patterns and 12 syzygy-problems invented for this research to measure the ability to play Syzygies. The syzygy-problems were implemented once a week. To motivate participants to the game we have added another point to the scoring rules.

(8) The sum of twice (7) plus:
3 points - if the chain has 2 links;
2 points - if the chain has 3 or more links.

The point (8) establish that a participant have at least 2 points if the chain is correctly done.

The statistical analysis carried out on data revealed a correlation between the ability to play Syzygies and factor five ($r =,62; p <,01$). We have also analyzed data using the scores of the Lewis Carrol scoring rules (without adding point (8)) and the coefficient of correlations was ,59.

		Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7
Syzygies	Pearson Correlation	,255	,014	,092	,174	,617**	-,252	-,333
	Sig. (2-tailed)	,307	,955	,716	,489	,006	,314	,177
	N	18	18	18	18	18	18	18
Syzygies L.C.	Pearson Correlation	,168	-,017	,146	,010	,592**	-,255	-,368
	Sig. (2-tailed)	,506	,946	,563	,968	,010	,307	,133
	N	18	18	18	18	18	18	18

**. Correlation is significant at the 0.01 level (2-tailed).

Figure 3: Correlation between Syzygies scoring and the seven factors

Accordingly to the results obtained in the analysis, we conclude that for 4^{th} grade students there is a relationship between the ability to play Syzygies and the ability to find patterns involving odd and even numbers. In others words, this word game has a connection with patterns involving odd and even numbers for this students.

These results are important in our research by the fact that is a step forward in the search of the comprehension of the relationship between the ability to play different patterns and the ability to play different games.

References

Cohen, L., & Manion, L. (1989). *Research Methods in Education* (3rd Ed.). London: Routledge.

DEB (2001). *Currículo nacional do ensino básico.* Lisboa: Editorial do Ministério da Educação.

Devlin, K. (1997). *Mathematics: the science of patterns.* New York: Scientific American Library.

DGIDC (2007). *Programa de Matemática do Ensino Básico.* Documento recuperado em 04/01/2008 em `http://sitio.dgidc.min-edu.pt/PressReleases/Paginas/ProgramadeMatematicadoEnsinoBasico.aspx`

Ferreira, D., & Palhares, P. (2007). *O jogo de xadrez e a identificação de padrões.* Boletim da SPM, 56, pp. 93-112.

Ferreira, D., & Palhares, P. (2008). *Chess and problem solving involving patterns.* The Montana Mathematics Enthusiast. Vol. 5 n. 2 & 3. pp. 249-256.

Ferreira, D., Palhares, P., & Silva, J. N. (2008). *Padrões e jogos matemáticos.* REVEMAT - Revista Electrônica de Educação matemática, v3.3, pp. 30-40, UFSC.

Ferreira, D., Palhares, P., & Silva, J. N. (2010). *Mathematical skills and mathematical games.* In Jorge Nuno Silva (Ed) Proceeding of the Recreational Mathematics Colloquium I (pp. 89-94). Lisboa: Ludus.

Field, A. (2000). *Discovering Statistics using SPSS for Windows*. London: Sage publications.

Fraenkel & Wallen (1990). *How to Design and Evaluate Research in Education*. New York: Mc Graw-Hill.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: Chicago University Press.

Murray, H. J. R. (1952). *A History of Board-Games other than Chess*. Oxford: Clarendon Press.

Neto, J. P., & Silva, J. N. (2004). *Jogos matemáticos, Jogos Abstractos*. Lisboa: Gradiva.

Posamentier, A. S., & Krulik, S. (1998). *Problem-solving strategies for efficient and elegant solutions: A resource for the mathematics teacher*. Thousand Oaks: Corwin Press, Inc.

Steen, L., A. (1990). *On the Shoulders of Giants: New Approaches to Numeracy*. Washington: National Academy Press.

Wakeling, E. (1995). *Rediscovered Lewis Carroll Puzzles*. New York: Dover Publication.

Whitehill, B. (2009). *Toward a Classification of Non-Electronic table Games*. In Jorge Nuno Silva (Ed). Proceeding of Board Games Studies Colloquium XI (pp. 53-66). Lisboa: Associação Ludus.

Mathematics in *Cartoons*: a brief journey

Natália Bebiano, *Jason N. Bolito*, *F. J. Craveiro de Carvalho*
Departamento de Matemática
Universidade de Coimbra
3000-451 Coimbra
PORTUGAL

In rough terms, a cartoon can be described as consisting of words and a drawing. Words are sharp, ironical or tragic, but they can be dispensed with if the drawing is self-sufficient. The quality of the drawing is important, though not essential.

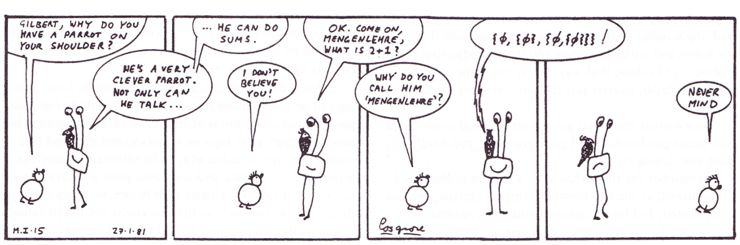
Almost every newspaper publishes cartoons, very often as a form of immediate political commentary, but they appear in several other contexts, namely in the scientific one.

In this article we propose an analysis of some of the works by cartoonists who used non-trivial Mathematics.

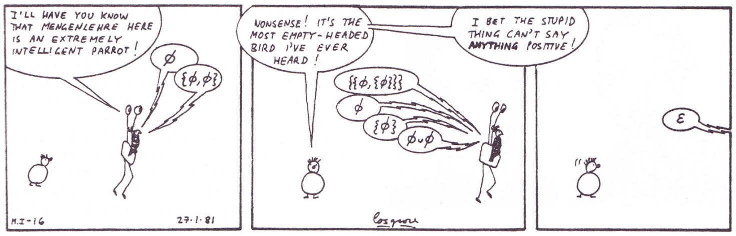
Cosgrove

In some of the first issues of *The Mathematical Intelligencer* and *Manifold*, the latter a publication at Warwick University, U. K., which would come to give origin to *Seven years of* Manifold, we find several cartoons signed by Cosgrove. Cosgrove was a pseudonym used by mathematician Ian Stewart, also famous by his highly active role in mathematical popularization.

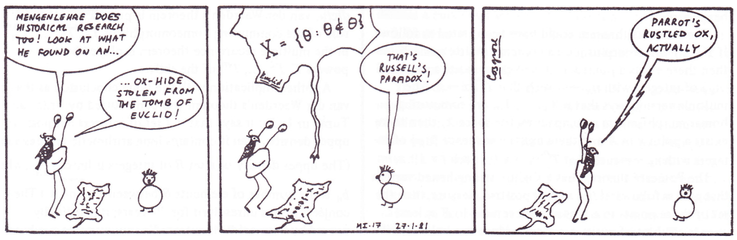
One of his *heroes* is the parrot *Mengenlehre*, a specialist in set theory particularly able when manipulating the empty set, $\varnothing$. Such an ability enables him, for instance ("The Mathematical Intelligencer," 1982), to do sums (Halmos, 1974).

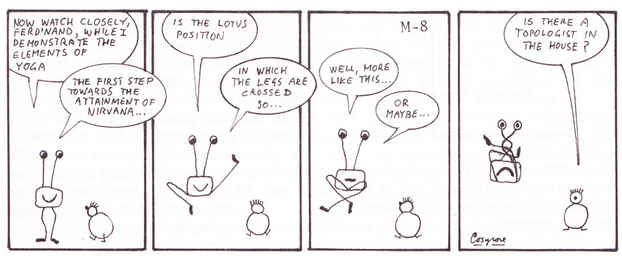


But that is not all. For example, in ("The Mathematical Intelligencer," 1982), there is a subtle to allusion ε and δ definitions.



Another character is *Ferdinand*, who can identify *Russell's Paradox* ("The Mathematical Intelligencer," 1982) and is familiar with some Topology areas, Knot Theory (Stewart & Jaworski, 1981) being an example.





Sidney Harris

Sidney Harris is a prolific cartoonist, author of several books of Science cartoons (Sciencecartoonsplus). The quality of his drawing is excellent and, on his work, double Nobel Prize Laureate Linus Pauling said:

- *By my criterion of humor, whatever it may be, Harris is successful about 99 % of the time.*

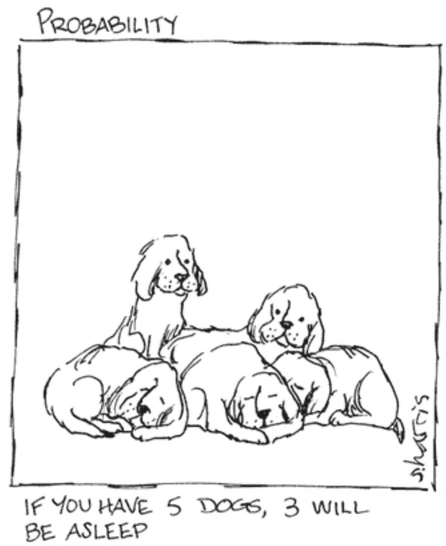
One of his cartoons shows the word *Probability* at the top, 5 dogs, 3 of which are asleep, and the caption - *If you have 5 dogs, 3 will be asleep.*

The claim is obviously false, we may have 3 dogs awake and 2 asleep. Harris may mean that there will always be 3 dogs asleep or 3 dogs awake, having chosen *asleep* possibly to refer to the laziness of the animals.

We are inevitably led to think of the following result:

In any group of, at least, 6 people there are 3 who know one another or 3 who are strangers to each other.

This is a result of *Ramsey Theory* and, to prove it, it is enough to consider a 6 element set (Jones, 2000). However the result is false if we have only 5 people.



The interpretation we give of the Harris cartoon can however be easily justified.

Choose a dog. Suppose it is awake and call it A. Then there must be 2 dogs who are asleep or 2 dogs who are awake. Hence we have

$$\mathrm{A} \;—\; \mathrm{A} \;—\; \mathrm{A}$$

or

$$\mathrm{A} \;—\; \mathrm{S} \;—\; \mathrm{S}$$

In the first case, we already have 3 dogs awake. As to the second, the 2 other missing dogs are both asleep,

$$\mathrm{A} \;—\; \mathrm{S} \;—\; \mathrm{S} \left\langle \begin{array}{l} \mathrm{S} \\ \mathrm{S} \end{array} \right.$$

one is asleep, the other is awake, we have

$$\mathrm{A} \;—\; \mathrm{S} \;—\; \mathrm{S} \left\langle \begin{array}{l} \mathrm{S} \\ \mathrm{A} \end{array} \right.$$

or are both awake.

$$\mathrm{A} \;—\; \mathrm{S} \;—\; \mathrm{S} \left\langle \begin{array}{l} \mathrm{A} \\ \mathrm{A} \end{array} \right.$$

In the first two hypotheses we have 3 dogs asleep and, in the last one, we have 3 dogs awake. Instead of starting with a dog A who is awake, we might similarly start with a dog asleep S. Therefore the result is proven.

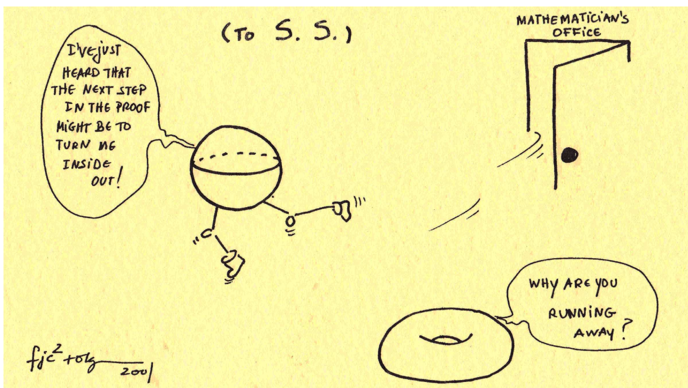
A theorem of Steve Smale

At the end of the 1950's, while a research student under Raoul Bott's supervision, Steve Smale proved a result, known as *"the eversion of the sphere"*, which, without getting into technical details, can be stated as follows:

The inclusion i of the euclidean sphere S^2*, having centre at the origin and with radius 1, into* R^3 *can be deformed into the antipodal map* $A : S^2 \to R^3$*, which assigns* $-x$ *to* x*.*

The deformation is given by a family $f_t, t \in [0, 1]$, of maps, with $f_0 = i, f_1 = A$, which do not have to be injective, but are such that no crease is created.

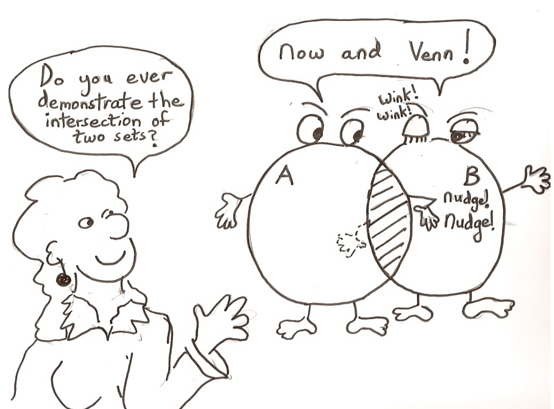
Smale showed the existence of such a family, but not how to get it. The first example of visualization of the procedure is due, among others, to Arnold Shapiro and the french Bernard Morin (Phillips, 1966). Morin, born in 1931, is blind since he was six.



Tom Canel

Of all the cartoons which are considered in this article, some of the mathematically deeper are by Tom Canel (Bligblug). It is a pity his drawings are not better and that the lettering is not a bit more professional.

In the mentioned *blog* (Bligblug) we find, for example, a cartoon where the pun between *now and then* and *now and Venn*, Venn from *Venn diagram*, is explored.

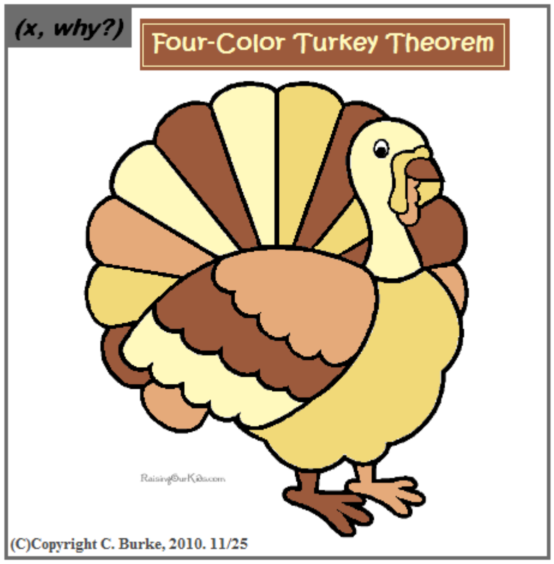


In another one, the expression *to lose face*, which means to be humiliated, is interpreted literally, up to a homeomorphism. When we form the connected sum of two tori, we remove an open disc from each one and glue the remaining parts by the borders that were created. Now an open disc is homeomorphic to the interior of any face in a torus triangulation.

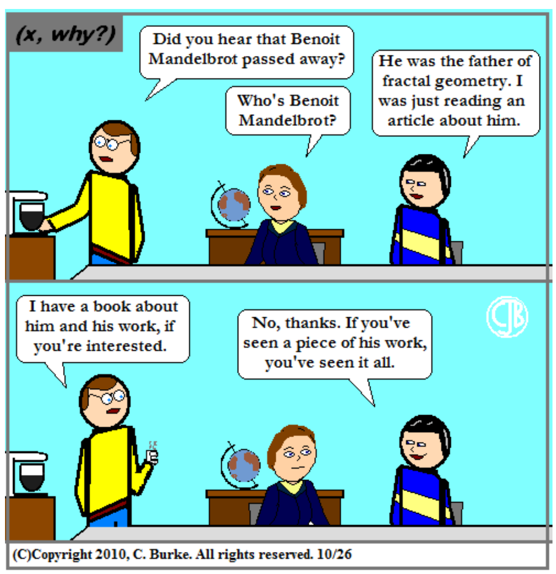
In a purely algebraic example, Canel uses the fact that word *field* means simultaneously an algebraic structure and a plot a land.

Christopher Burke

In the work of Christopher Burke (Mrburkemat) we find, for instance, a map drawn on a very Christmas turkey, coloured with just four colours. The cartoon is appropriately titled *Four-Color Turkey Theorem*. The use of capital letters adds to the ambiguity by relating bird and country.



On the death of Benoît Mandelbrot, Bruce drew a cartoon



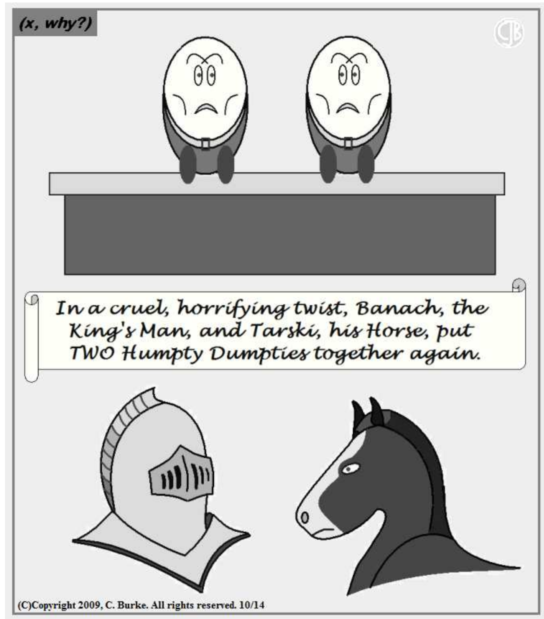
where the final retort is:

- *If you've seen a piece of his work, you've seen it all.*

a clear reference to the nature of fractal objects.

A much less obvious cartoon is one where we have two oval figures and where, in the caption, Banach, Tarski and two Humpty Dumpties are mentioned. In fact, the caption reads

In a cruel, horrifying twist, Banach, the King's man, and Tarski, his Horse, put TWO Humpty Dumpies together again.



Banach and Tarski suggest right away the Banach-Tarski paradox. According to Wikipedia,

The BanachTarski paradox is a theorem in set theoretic geometry which states that a solid ball in 3-dimensional space can be split into a finite number of non-overlapping pieces, which can then be put back together in a different way to yield two identical copies of the original ball. The reassembly process involves only moving the pieces around and rotating them, without changing their shape.

and, quoting the same source again,

Humpty Dumpty is a character in an English language nursery rhyme, probably originally a riddle and one of the best known in the English-speaking world. He is typically portrayed as an egg and has appeared or been referred to in a large number of works of literature and popular culture.

the more common nursery rhyme version being

Humpty Dumpty sat on a wall,

Humpty Dumpty had a great fall.

All the king's horses and all the king's men

Couldn't put Humpty together again.

The cartoon then becomes easy to read.

The authors gratefully thank Professors Ian N. Stewart and Christopher J. Burke and Dr Tom Canel for the permission to include their work. As for Sidney Harris, despite continuous efforts, they did not have any reply to their contact attempts.

References

Bligblug. http://www.bligblug.blogspot.com

Halmos, Paul R. (1974). *Naive Set Theory*, Springer-Verlag.

Jones, Gareth A. (2000). PAUL ERDÖS*: A brief introduction to his mathematical achievements*, Departamento de Matemática da Universidade de Coimbra.

Mrburkemath. http://mrburkemath.blogspot.com

Phillips, Anthony (1966, May). *Turning a surface inside out*, Scientific American.

Sciencecartoonsplus. http://www.sciencecartoonsplus.com

Stewart, Ian, & Jaworski, John (Eds.). (1981). *Seven Years of* Manifold*, 1968-1980*, Shiva Publishing Limited.

The Mathematical Intelligencer. (1982). Vol. 4.

Mathematical Quilts

Andreia Hall
Mathematics Department
University of Aveiro
Portugal
andreia.hall@ua.pt

Abstract

Mathematics play an important role in quilt making. Tiling, symmetry, fractals, rep-tiles and Voronoi diagrams are just a few of many mathematical concepts that can be used and explored in patchwork. In this talk we shall present some quilt examples that use mathematical models.

Introduction

Recreational mathematics is broad term, usually referring to mathematical puzzles and mathematical games. But many other topics can be included under the umbrella of recreational mathematics as for instance fractals and origami. In this paper we present some applications of Mathematics to quilting and patchwork.

Patchwork or "pieced work" is a form of needlework that involves sewing together pieces of fabric into a larger design. The larger design is usually based on repeated patterns built up with different colored shapes. Patchwork is most often used to make quilts and some of them are often combined with embroidery and other forms of stitchery.

When used to make a quilt, the patchwork or pieced design becomes the "top" of a three layered quilt, the middle layer being the batting, and the bottom layer the backing. Quilting is the sewing method (by hand or sewing machine) done to join the layers.

Many patchwork designs are based on tiling properties and this is where some mathematics may come in. Patterns and symmetry usually go together in this type of work. Other patchwork designs may be based on other types of mathematical models such as fractals, spirals and various types of diagrams.

In this work we present one example of a patchwork design which deals with tiling and symmetry, another example of a designs which is based on a recursive tiling property (reptiles), two examples of designs which are based on fractals and two pairs of pieces which are based on Voronoi diagrams.

Many of the models used in the construction of these works were taken from Wolfram's Domonstrations Project website. This site provides many interesting and interactive modules which may be very useful as a source of inspiration for quilt designs.

Patchwork design based on tiling with right isosceles triangles

Figure 1 presents a picture of a quilt which is based on a simple square grid, each square being divided in two triangles across one of the diagonals. The emerging pattern respects a double layered type of symmetry: each pair of triangles that form a square are made of "symmetrical" fabrics in the sense that fabric pattern is the same but the colours switch. At a more global scale, the whole work contains a symmetry axis along one of the diagonals of the large square, however this is not a pure symmetry, it is rather a sort of inverted symmetry - the lines are reflected but the colours are inverted.

Figure 1: Red and white symmetry.

Patchwork designs based on Rep-tiles

Rep-tiles, or Replicating Tiles, are tiles that can be joined together to make larger replicas of themselves. The term "rep-tile" was coined by mathematician Solomon Golomb, a prolific contributor to geometrical recreations of all sorts. Rep-tiles that require n tiles to build a larger version of themselves are said to be rep-n. For example, we can combine four squares to make a bigger square, so a square is rep-4.

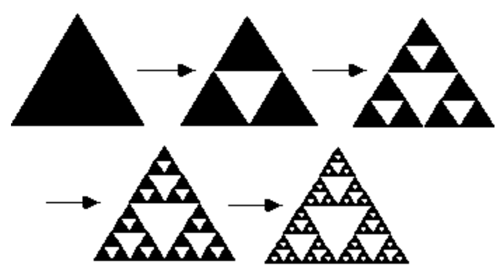
Figure 2 contains an example of a quilt design based on rep-tiles. In this case the basic figure is a right triangle with edges 1 and 2 units long. This triangle is rep-4 and rep-5 and in this work it is explored as a rep-5.

Figure 2: Rep-tiles.

This work used stripped fabrics with various colours and some texture was added by placing metallic pins all over it.

Patchwork designs based on fractals

The mathematical model underlying the piece of work in Figure 3 is the Sierpinski's Triangle, after the Polish mathematician Waclaw Sierpinski who described some of its interesting properties in 1916. Sierpinski's triangle is one example of a fractal structure based on the repeated division of an equilateral triangle into 4 smaller ones as shown in the picture on the right. This division is performed infinitely many times however, in this work we only performed the first 5 iterations.



Sierpinski's Triangle has been used by many artists. We recreated the triangle through patchwork, quilting and appliqué, using floral designs which contrast with plain black fabric along the recursion. In one of the iterations we also used three repetitive plain colours which contrast with the free floral designs of the other iterations. Finally the triangle was freed from its rigidity through appliqué stretching throughout the whole work.

Figure 3: Sierpinski's triangle revisited.

Figure 4 contains another example of a fractal design based on a square.

Figure 4: Trisection of the square.

In this case the square is divided into four similar squares, three of which are kept without any more transformation. The fourth square is divided again using the same strategy. This procedure was repeated seven times. The fabrics used along the iterations alternate between red and blue colours giving some dynamics to the resulting design.

Patchwork designs based on Voronoi diagrams

Voronoi diagrams are a special kind of decomposition of the plane determined by distances to a specified discrete set of points (called Voronoi sites). The plane is divided into regions (cells) determined by the smallest distance to the Voronoi sites. These diagrams are named after Georgy Voronoi but sometimes they are called Dirichlet tessellations (after Lejeune Dirichlet).

Figure 5: Floral Voronoi I and II

Figure 5 shows a pair of quilts based on Voronoi diagrams built in such a way that they resemble floral pictures. In both works the quilts were creatively explored through hand sewing and application of other materials on the Voronoi sites.

Figure 6 contains another pair of quilts based on Voronoi diagrams now resembling sun images.

Figure 6: Solar Voronoi I and II.

References

Hall, A., & Leite, P. (2010). *Floral Voronoi I and II.* Bridges Pécs, Art Exhibition Catalog 2010. In Robert Fathauer and Nathan Selikoff, editors, Tessellations Publishing, p.8.

Dutch, S., `http://www.uwgb.edu/dutchs/symmetry/reptile1.htm`.

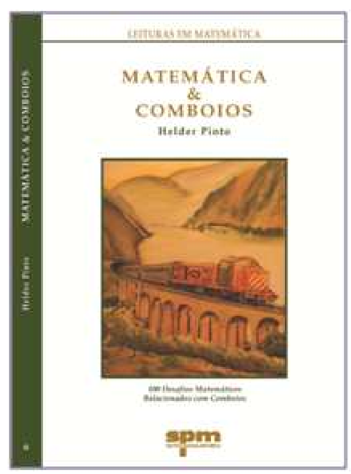
Wolfram. `http://demonstrations.wolfram.com/`

Some math problems with trains and railways

Helder Pinto
FCUL
hbmpinto1981@gmail.com

One of the most important teacher's tasks is to find problems that are appealing to their students. There are many interesting problems/puzzles that are true classics (for example, the riddles of Sam Loyd, Henry Dudeney and Lewis Carroll), but do not always have a sufficient appealing and attractive context to today students - although they remain current and relevant, in their mathematical aspects, as the day they were created. However, with a little imagination and work, these problems can be adapted to different contexts, more modern as, for example, transports, sports, and the everyday life.

The talk presented showed some examples of well-known problems/puzzles which have been adapted to trains and railways (theme, so the author has found, that seems to be part of the imaginary of everyone, whether kids or grown-ups) and published on a book by the Portuguese Society of Mathematics (*Matemática & Comboios*, 2010). The choice of this particular theme was due to the author's own interest in this subject and tries to show that the same mathematical problem/puzzle may have various formulations in the quest to find something that is truly meaningful and interesting to the audience (usually children and adolescents) that is intended.



This book consists of 100 puzzles/riddles that have in common the use of the railway theme like, for instance, trains, stations, railway lines and timetables. From my personal experience, the trains cause a certain fascination in almost all people (which boy has never had fun with a toy train?). Math causes the same kind of fascination, but sadly, in a smaller number of people. Throughout my life, I have found many people who hated math (I say too many...) but thought curious my enthusiasm for trains and railways. From this fact arose the idea of writing a book that would combine these two realities: the trains and mathematics. In order to be an attractive book to all, the choice fell in the adaptation of some mathematical puzzles to the railway theme:

- some are published as they were originally written;

- many of them are adapted from classical riddles that use other themes and contexts;

- some are completely new.

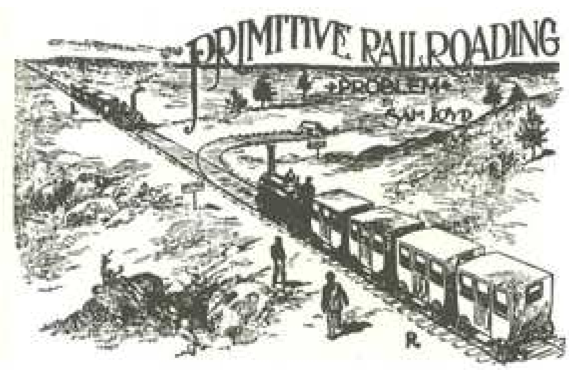
The classical authors in this book include such remarkable names as Brian Bolt, Henry Dudeney, Martin Gardner, Sam Loyd, Mariano Mataix, Yakov Perelman, Dennis Shasha, Malba Tahan, Eduardo Veloso e José Paulo Viana - these last two authors are well-known in Portugal for their weekly column of mathematical riddles in “Público” (Portuguese newspaper). This book uses, in general, the Portuguese reality (many of the lines, trains and pictures on this book are real and can be found in Portugal). For example:

- the Douro, Tua, Sabor, Corgo, Cascais and North Lines;

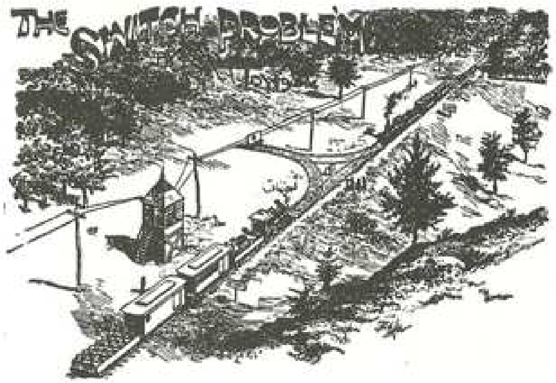
- the stations of Porto-Campanhã, Régua, Pocinho, Barca D’Alva, Leixões, Coimbra, Tomar and Rossio;

- the Sud Express, the Interrail, the Port Wine, the touristic trips and other material from Portugal.

The railway theme has been used a few times in the creation of mathematical riddles. For example, Sam Loyd created two well-known problems about the crossing of two trains with the only help of a small auxiliary line (just enough to contain a carriage or, alternatively, a locomotive). In the first puzzle we have two trains (consisted by one locomotive and several coaches) who want to pass by each other, with the additional restriction of not being possible to engage any carriage to the front of any of the locomotives.

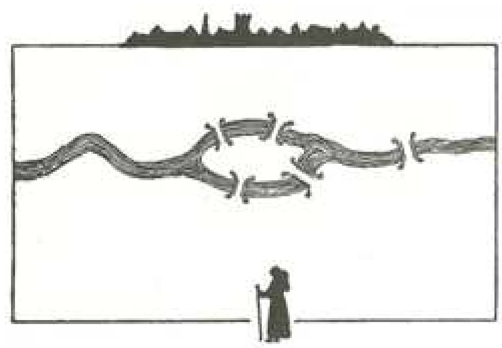


In the second, a train intends to pass through another train whose locomotive is damaged. The foregoing restriction is not applied to this situation: the locomotive that is not damaged can engage the carriages both in front and rear (the defective locomotive works like any other coach in this issue).

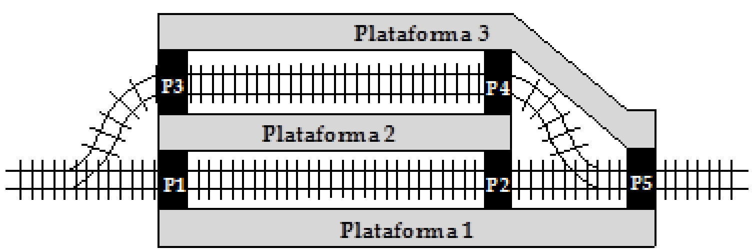


In the next pages, several other mathematical riddles will be presented as well as their adaptation to the railway theme.

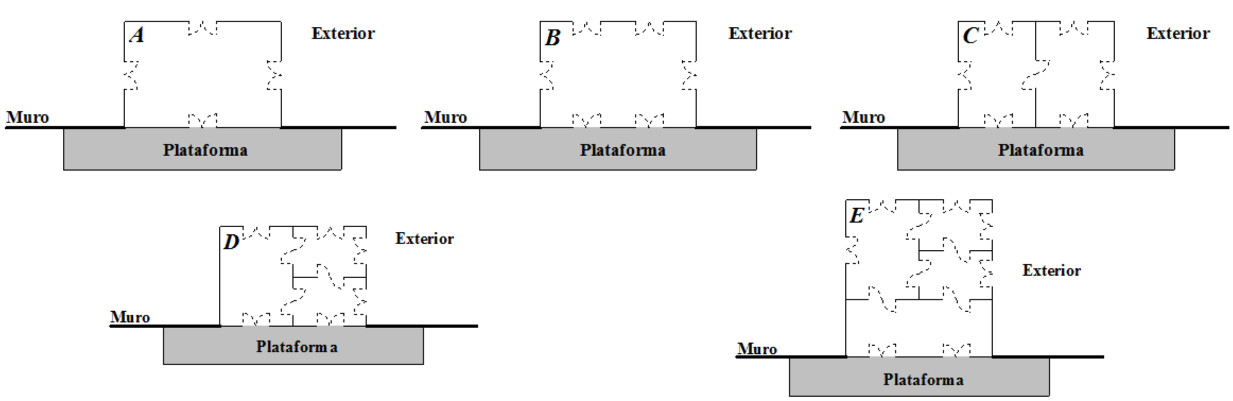
Many classical riddles involve paths such as, for instance, the problem of the seven bridges of Konigsberg. In general, in this kind of problem we have one of these two questions: identify a tour (if it exists) that obey to certain conditions or determine how many different possibilities exist for a ride as pretended. For example, Dudeney proposed the following scheme of a river with an island and five bridges, where a monk is on the opposite side of his monastery.



The question raised was how many different ways the monk can take to go to his monastery crossing all the bridges only once. Adapting it to the railroad theme, a simple possibility is to consider a station with three platforms, with the passages between them marked on the picture below (P1 to P5), which puts us in a situation entirely analogous to the Dudeney's problem.

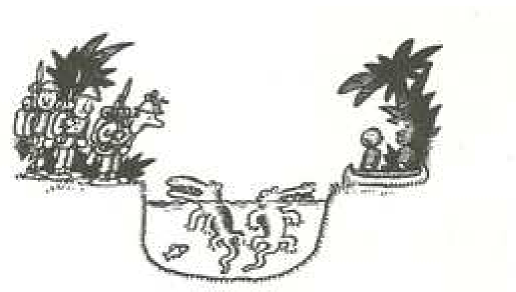


A somewhat similar problem to the last one is the following (adapted from Veloso and Viana): suppose that in the figure below, the plants (representing the walls, the doors and the access platform to trains) of some railway stations are presented, marked A through E.

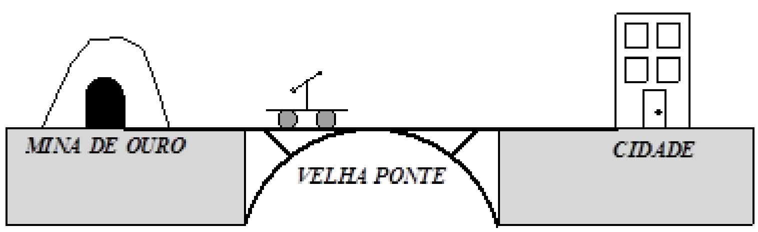


The question now is to determine in which stations it is possible for an employee, who intends to inspect the operation of all doors, to do it passing only once on each door and starting and ending its journey outside the station.

Another type of classical riddles consists in problems of crossing to the other side (generally, of a river) under several restrictions. One example is proposed by Bolt: suppose that an army pretends to cross a river infested with crocodiles; their only help are two native boys and one canoe that cannot travel with more than one soldier and his belongings or, instead, the two boys.



The question now is how to cross all the soldiers to the other side of the river. One possibility of adaptation of this problem to the railway theme is taking an old limited bridge to be crossed (in order to reach the city) by several miners with their gold and they only have a small pump trolley (railroad car powered by its passengers) and two persons in the city to help them. Knowing that the bridge is limited to the weight of two men without gold or, instead, one man with his gold, how to cross all the miners and their gold to the city?



Another riddle using old limited train bridges is the following: one bridge is limited to 13,5 tones (no crossing by foot is allowed); how to cross three locomotives with 6, 7 and 8 tones by one single man?

Some logical problems and riddles that include one or more liars are very interesting and, in general, they are easily adapted to the railway theme. For instance, suppose that a train has six security components showing the following information:

Security systems of a train		
A Component	**B Component**	**C Component**
C is broken	A is broken C is not broken	E is broken D is not broken
D Component	**E Component**	**F Component**
F is not broken	C is broken F is not broken	B is broken

Is it possible that just one component is broken? Which components of this train are damaged if the problem is only in two of them?

Another simple example of a problem with liars: A man asked to a group of four boys the following question:

– Who entered the train without a ticket?

Their answers were:

– I wasn't - said António.

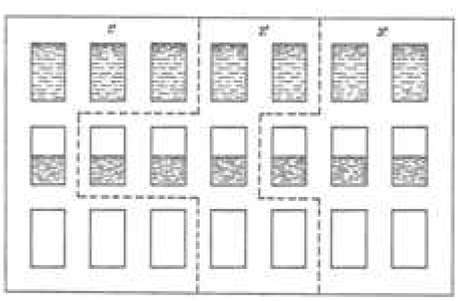
– It was Duarte - Bruno.

– It was Bruno - Carlos.

– Carlos is lying - Duarte.

Which is the answer to the man's question if there is just one liar within this group?

The solution of the well-known "21 casks of wine" problem, presented in the Tahan's book *The Man Who Counted*, is represented in the next diagram. In this riddle, 21 identical casks of wine of which 7 are full, 7 are half full, and 7 are empty, need to be divided equally among three men, *i.e.*, each man should receive the same amount of wine and the same number of casks (it is not possible to transfer wine from one cask to another). How to solve this?



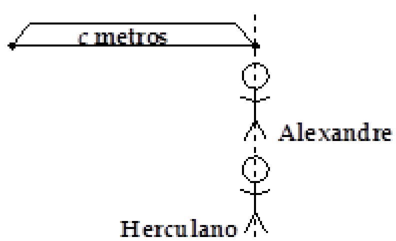
On the book *Matemática & Comboios* this riddle is presented replacing the casks by wagons and the wine by freight containers - each full cask is replaced by a wagon loaded with two containers and each half full is replaced by a wagon loaded with one single container.

The problems related to velocities are "mandatory" when the subject in question is vehicles (cars, bus, bikes, airplanes...) and the trains are no exception. It is possible to construct many interesting puzzles playing with the properties of time and the definition of velocity. In the following problems it is always assumed that the trains travel with constant velocity and the acceleration time is negligible. One very meaningful example, adapted from one presented by Gardner, is this: a train should go to a city in a round trip (one way and return) with an average speed of 60 km/h. Since the train was quite heavy, it only achieved an average speed of 30 km/h in the way to the city. At what speed the train needs to make the return journey to reach the desired average speed (the answer may be surprising...)?

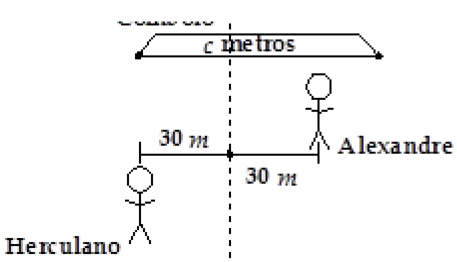
In another riddle we have the following information about one train that will travel between two stations (without in-between stops) and it is asked at what time is the departure of this train and at what speed should it go to arrive to the destination station at 12.30 (it's not 100 km/h ...).

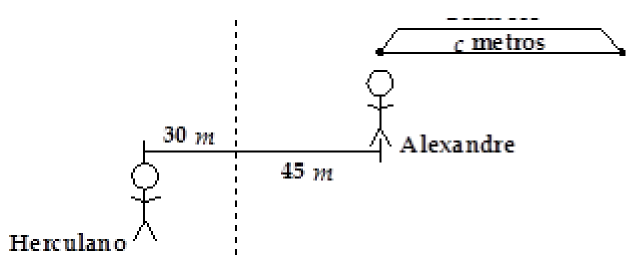
Possible velocities for the train	Arrival time on the destination station
80 km/h	13.00
120 km/h	12.00

Another very attractive puzzle, involving one train and two boys standing in a station, was presented in the Portuguese Math Olympics in 2006. In this, we have two friends, Alexandre and Herculano, that start walking (at the same velocity and in opposite directions) when the front of a train passes by them.



Each of them stopped when the rear-end of the train passes by them; Herculano walked 30 m and Alexandre 45 m. With this data, it was simply asked the length of this train.





A very important problem in our days is the "transportation or distribution problem". A typical transportation problem deals with sources where a supply of some product is available and destinations where the product is demanded. In general, we have different prices of transportation from one source to each destination and the final goal is to minimize the total cost of transportation (this is a real and significant problem in many companies). This problem appears in many contexts with different products and we can choose, for instance, trains that need to be redistributed for several stations. Suppose that the stations A and B have more trains than necessary and there is a lack of trains in the stations X, Y and Z, according to the following tables.

Station	Number of remaining trains
A	10
B	5

Station	Number of missing trains
X	8
Y	3
Z	4

Assuming that the cost of changing each train between two stations is represented in the next chart, determine the cheapest way of changing the 15 trains.

	X	***Y***	***Z***
A	70 €	40 €	30 €
B	30 €	10 €	20 €

Another topic very used in classical riddles is money. These kinds of problems are easily adapted to vary contexts, including the railway theme, as we can see in the following examples:

In one city there was a raise of the train tickets price in the middle of one month; three friends have the following chat:

– This month I made 12 trips by train and spent 14 €.

– I spent the same money as you but only travelled 10 times by train.

– I spent the same 14 €, as you both, but I managed to make 14 trips.

Which was the price ticket before and after the increase of prices? (adapted from a Tahan's riddle)

The Transport Minister decided for a 10 percent increase in the price of all rail tickets. The clients protested and it was decided to cut 10 percent of the new fares. When is the price of tickets lower: before or after? (adapted from a Perelman's riddle)

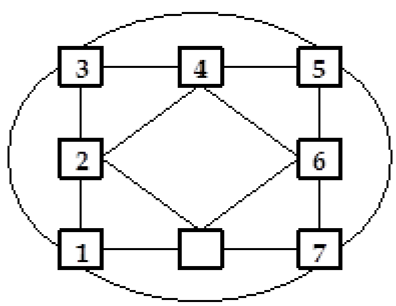
A boy had about 15 € in his pocket, only in coins of 1.00 € and 0.20 €. After buying his train ticket he had 1/3 of the original money and:

- So many coins of 1.00 € as he had 0.20 € originally;

- So many coins of 0.20 € as he had 1.00 € originally;

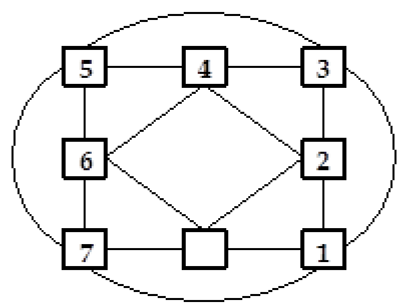
How much money did he spend in the train ticket?

Other very common type of riddle consists of changing the position of several objects obeying to some restrictions and minimizing the number of movements. The next two problems were originally proposed by Bolt and Dudeney and were adapted to the railway theme. The first is about 9 stations and 8 trains that have to change their position - see the scheme below to see the possible train routes and the initial and final configuration. Note that each train can only go to an empty station, *i.e.*, that don't have any train on it.



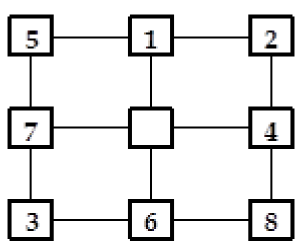
Initial

→



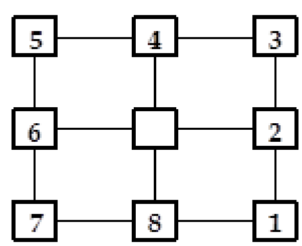
Final

The second involves only 7 trains and 8 stations since the train number 5 is broken and, because of that, can't be moved out from its station.



Initial

→

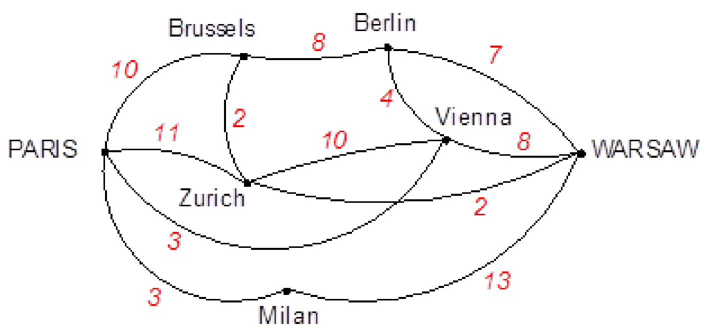


Final

Another well-known problem is "the travelling salesman problem": given a list of cities and their pairwise distances, the task is to find a shortest possible tour that visits each city exactly once. Another common version of this problem, with the same initial data, is related to the construction of roads or railways and it is inquired how to construct the cheapest network connecting all the cities. The book *Matemática & Comboios* uses the planning of a new subway system of a city and the pairwise costs (next table) of connecting the places that are intended to be connected by this mean of transportation.

(EUR million)	City Hall	Stadium	Hospital	Beach	Cinema	University
City Hall	-	-	-	-	-	-
Stadium	7	-	-	-	-	-
Hospital	15	10	-	-	-	-
Beach	8	5	9	-	-	-
Cinema	8	11	13	6	-	-
University	20	14	4	12	16	-

Another riddle (adapted from Shasha) involving railway networks is now presented. Many fans of football from Paris intend to go by train to the final of Euro 2012 in Warsaw. Because of high demand, there are only a restricted number of tickets for each possible journey that they can choose - the numbers of tickets available for each of the represented journeys are presented below.



With these limitations, how many people can travel between Paris and Warsaw by train?

At this point, after all these examples, the important message to pass is that the same problem can have various formulations (more or less appealing depending, largely, on the reader's preferences). For instance, consider the following problem (adapted from Perelman).

The Railway Company needs to print 9000 schedules. One (very modern) machine takes two hours to do the service; the other takes three hours (because it is older). Using the two machines simultaneously, how quickly can we do the requested service?

Perelman used the case of two secretaries typing a text and it is possible to choose many others formulations to this problem, such as: two men painting a house, two machines doing the same toy, two tractors plowing a field, two factories producing the same drink, two brothers mowing the lawn or two computers processing data. The possibilities are almost endless...

Some final considerations:

- There are many interesting problems/puzzles that are true classics (for example, the riddles of Sam Loyd, Henry Dudeney and Lewis Carroll) that can be used nowadays;

- Many of those problems could be adapted to new contexts; with a little imagination and work, these problems can be adapted to different contexts, more modern as, for example, transports, sports, and the everyday life.

- It is important to choose themes that are truly meaningful and interesting to the target audience (usually children and adolescents);

- By my experience, everyone likes trains (seems to be part of the imaginary of everyone, whether kids or grown-ups); so, I think that these problems could be interesting for a vast range of people;

- I chose trains because I like them; others could prefer other themes; the important is choosing "whatever works" (more tools available, more chance of success in attracting people to maths).

References

Bolt, Brian (2008). *Actividades Matemáticas*, RBA Coleccionables.

Bolt, Brian (2008). *Uma Paródia Matemática*, RBA Coleccionables.

Dudeney, Henry (2008). *Os Gatos do Feiticeiro*, RBA Coleccionables.

Gardner, Martin (1993). *Ah, Apanhei-te!*, Gradiva, Lisboa.

Kruskal, Joseph (1956, Fevereiro). *On the shortest spanning subtree of a graph and the travelling salesman problem*, Proceedings of the American Society, vol. 7, n.º 1.

Loyd, Sam (2008). *Os Enigmas de Sam Loyd*, RBA Coleccionables.

Loyd, Sam (2008). *Os Novos Enigmas de Sam Loyd*, RBA Coleccionables.

Perelman, Yakov (1979). *Matemáticas Recreativas*, Litexa, Lisboa.

Perelman, Yakov (2008). *Experiências e Problemas Recreativos*, RBA Coleccionables.

Pinto, Helder (2010). *Matemática & Comboios*, Sociedade Portuguesa de Matemática.

Shasha, Dennis (2008). *As Enigmáticas Aventuras do Dr. Ecco*, RBA Coleccionables.

Tahan, Malba (2001). *O Homem Que Sabia Contar* (3.ª ed.), Editorial Presença, Lisboa.

Veloso, Eduardo, & Viana, José Paulo (2008). *Árvores e Castelos*, RBA Coleccionables.

XXIV Olimpíadas Portuguesas de Matemática, Final (1.º dia), Categoria A. 31 de Março 2006.

Winning Nim with Beatty and Fibonacci

M. J. Torres *
CMAT, Departamento de Matemática e Aplicações, Campus de Gualtar
4700-057 Braga, Portugal
jtorres@math.uminho.pt

Abstract

In this paper we describe how to win Wythoff's Nim with Beatty and Fibonacci. The winning pairs for Wythoff's Nim are related with Beatty sequences. The winning strategy for Wythoff's Nim using the Fibonacci number system mimics the winning strategy for Bouton's Nim using the binary number system.

Nim

Quoting Martin Gardner, (Gardner, 1988), "*The game of Nim is one of the oldest and most engaging of all two-person mathematical games known today*". The game of Nim is an example of a *combinatorial game*, i.e., a game of pure strategy with no random elements in which two players take turns making moves until a winning position for one of the players is reached. The solution concept for this class of games is a *winning strategy* - a sequence of moves that forces a win, no matter what moves the opponent makes. In 1901, Charles L. Bouton of Harvard University analyzed Nim and gave its winning strategy in (Bouton, 1901), based on the binary number system. According to Richard J. Nowakowski (Nowakowski, 2008), some discussions about the origin of the name *Nim* refer a Chinese origin. However, since Bouton did his PhD in Leipzig it is presumable that the name of the game comes from the german *nimn* which means 'to take'. A version of Nim is played, and has a symbolic relevance, in the French New Wave film *L'Année Dernière à Marienbad* (1961).

Playing Nim

In Bouton's original paper (Bouton, 1901) the game of Nim is described as follows: *The game is played by two players, A and B. Upon a table are placed three piles of objects of any kind, let us say counters. The number in each pile is arbitrary, except that it is well to agree that no two piles*

* *This research was financed by FEDER Funds through "ProgramaOperacional Factores de Competitividade - COMPETE" and by Portuguese Funds through FCT - "Fundação para a Ciência e a Tecnologia", within the Project PEst-C/MAT/UI0013/2011.*

shall be equal at the beginning. A play is made as follows: the player selects one of the piles, and from it takes as many counters as he chooses; one, two, ..., or the whole pile. The only essential things about a play are that the counters shall be taken from a single pile, and that at least one shall be taken. The players play alternately, and the player who takes the last counter or counters from the table wins.

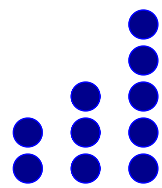


Figure 1: Three pile Nim.

The game just described can be at once generalized to the case of any number of piles, with the same rule for playing.

The first 'trick' from combinatorial game theory, which was discovered by Bouton, is how to win at Nim, using the binary system. Later, working independently, Roland P. Sprague (Sprague, 1935) in 1935 and Patrick M. Grundy (Grundy, 1939) in 1939 proved that every *impartial* game is equivalent to a Nim position (a game is *imparcial* when both players have the same moves). More recently, with the novel approach of *Winning Ways* (Berlekamp, Conway, & Guy, 2001), the theory of impartial games became a beautiful mathematical theory (see, for instance, (Santos, 2010)).

In the remain of this section we will learn how to win at Nim.

Winning Nim

We begin with the situation in which we have only one or two piles:

1. If there is only one pile, remove everything. You win the game!
2. If there are two piles of different size, remove from the larger to leave both with the same size, as in Figure 2. Then, just copy the moves of your opponent. You win the game!

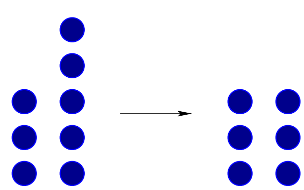


Figure 2: First player has the 'copycat' winning strategy.

Therefore, with one pile or two unequal piles the first player has a winning strategy. What about winning at Nim in the original description of Bouton with three piles? Let's try it! Suppose that we have three piles of sizes 2, 3, 5, as in Figure 1, and that we make the following 'trick': we remove four counters from the larger pile. It can be easily checked (see Figure 3) that whichever move the second player makes, the first can force two piles of equal size. Therefore, the first player has a winning strategy.

Figure 3: Winning Nim with a 'trick'.

What is the 'trick'? The 'trick' was discovered by Bouton (Bouton, 1901) and we will reveal it! The only ingredient we shall use is an algebraic operation, the *Nim-sum*, that was introduced by Bouton. We write the numbers of the counters in each pile in the binary scale notation, and place these numbers in three horizontal lines so that the units are in the same vertical column. Then we sum each column using addition modulo 2. For example, the Nim-sum of the first pile in Figure 3 can be calculated writing $2 = (10)_2$, $3 = (11)_2$ and $5 = (101)_2$ and adding modulo 2:

$$\begin{array}{cccc} & 0 & 1 & 0 \\ & 0 & 1 & 1 \\ + & 1 & 0 & 1 \\ \hline & 1 & 0 & 0 \end{array}$$

Figure 4: Nim-sum: $2 \oplus 3 \oplus 5 = (100)_2 = 4$.

Therefore, the Nim-sum $2 \oplus 3 \oplus 5$ is equal to $(100)_2 = 4$, which is different from 0. Let us now calculate the Nim-sum of the second pile in Figure 3, i.e., let us calculate $2 \oplus 3 \oplus 1$. Again we write the numbers of the counters in each pile in the binary scale notation, and place these numbers in three horizontal lines so that the units are in the same vertical column. In this case the Nim-sum is equal to zero!

$$\begin{array}{rcc} & 1 & 0 \\ & 1 & 1 \\ + & 0 & 1 \\ \hline & 0 & 0 \end{array}$$

Figure 5: Nim-sum: $2 \oplus 3 \oplus 1 = 0$.

Et voilà - this is the trick! A position, (x_1, x_2, x_3), in Nim is a winning position (for the previous player) if and only if the Nim-sum of its components is zero, $x_1 \oplus x_2 \oplus x_3 = 0$. This approach can be generalized to an arbitrary number of piles, say n, each with an arbitrary number x_n of counters. Bouton showed in (Bouton, 1901) how to win at Nim, using only the cardinality of the constituent piles. More precisely, Bouton proved that:

Theorem (Winning Nim with Bouton): *A position, $(x_1, x_2, \cdots, x_n)$, in Nim is a winning position if and only if the Nim-sum of its components is zero, $x_1 \oplus x_2 \oplus \cdots \oplus x_n = 0$.*

Wythoff's Nim

The Bouton paper stimulated the interest in the area of combinatorial games theory and suggested very interesting games. First was Wythoff's Nim, introduced and solved by Willem A. Wythoff (Wythoff, 1907), a PhD in Mathematics from the University of Amsterdam, in 1907. This game was previously played in China under the name of *tsyan-shidzi* ("choosing stones"), but was reinvented by Wythoff. The game was reinvented independently at least once more by Rufus P. Isaacs, a mathematician at Johns Hopkins University, in the 1960's. In this section we will describe the winning strategy for Wythoff's Nim. We will see how to win at Wythoff's Nim with Beatty (Coxeter, 1953) and how to win at Wythoff's Nim with Fibonacci (Silber, 1977).

Playing Wythoff's Nim

In Wythoff's Nim, *two piles of counters are placed on a table, the number in each pile being arbitrary. On a turn, a player can remove from one of the piles an arbitrary number of counters, as in Bouton's Nim, but has the added option of removing from both piles an equal number of counters. The player who takes the last counter or counters wins.* Another description of this same game was given by Isaacs about 1960 (see (Berge, 1962)): the game is played on a quarter infinite chess-like board with one chess Queen and a move must move the Queen closer to the corner of the board. The two piles of Wythoff's Nim can be interpreted as the coordinates of the Queen's position. Hence, this game is often called *Wythoff Queens*.

Wythoff's Nim is associated with interesting and beautiful mathematics, namely surprising connections to the golden ratio and Fibonacci numbers. In the remain of this section we will enjoy these beautiful connections, as we learn how to win at Wythoff's Nim.

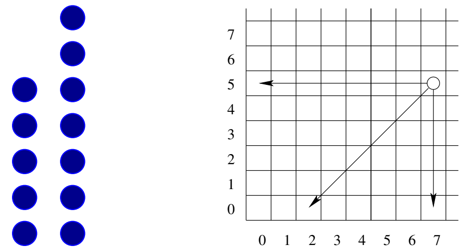


Figure 6: Wythoff's Nim as Whythoff Queens: the two piles of Wythoff's Nim can be interpreted as the coordinates of the Queen in Whythoff Queens. Diagonal move of the Queen is then equal to reducing the same number of counters from both piles, and horizontal or vertical move may be similarly represented by reducing the number of counters of one pile (specifically from the pile representing the direction of the move).

Winning Wythoff's Nim

The winning strategy for Wythoff's Nim is known since Wythoff's paper (Wythoff, 1907) and consists of always leaving our opponent one of a sequence of pairs,

$$(1,2),\ (3,5),\ (4,7),\cdots$$

which can be defined geometrically as follows: the terminal pair $(0,0)$ is a winning pair. Searching for the next winning pair of Wythoff's Nim (or Wythoff pair), it is clear that if we move the Queen to the bottom row or leftmost column, we are going to lose - our opponent will move the Queen to the origin next move. Similarly, if we move the Queen onto the diagonal. In fact, we can easily see that the closest Wythoff pair to the origin is the pair $(1,2)$ (or, $(2,1)$ which we shall identify). If we move the Queen to any of these pairs, our opponent can only move to $(1,1)$, $(1,0)$, $(0,1)$, $(2,0)$ or $(0,2)$. From any of these pairs we can move to $(0,0)$ and win the game. Proceeding the same way, see Figure 7, we get that the next Wythoff pairs are $(3,5)$, $(4,7)$, and so on.

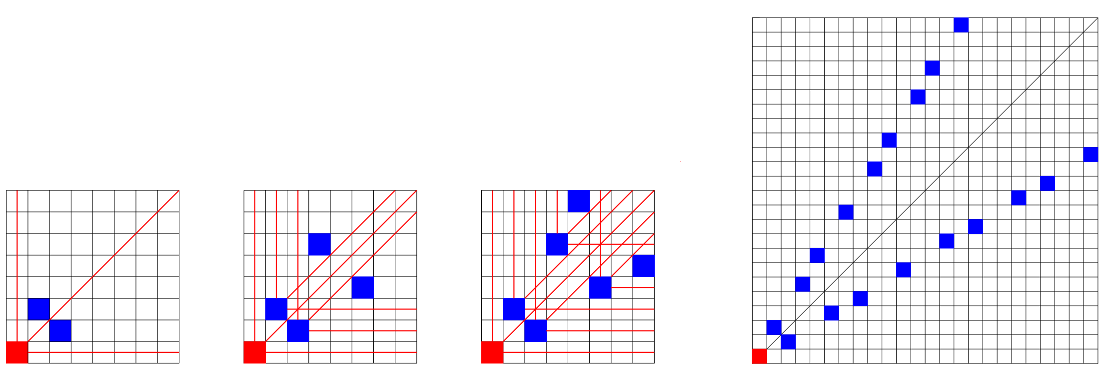


Figure 7: Wythoff pairs represented graphically.

Hence, a little reflexion shows that the Wythoff pairs can be defined inductively as follows: we begin with the pair $(a_1, b_1) = (1, 2)$, and, having defined (a_1, b_1), (a_2, b_2), ..., (a_n, b_n), the smaller number a_{n+1} of the next pair is the smallest positive integer not already used, and then the larger b_{n+1} is chosen so that the difference $b_{n+1} - a_{n+1}$ be $n + 1$.

n	1	2	3	4	5	6	7	8	$\cdots$
(a_n, b_n)	$(1,2)$	$(3,5)$	$(4,7)$	$(6,10)$	$(8,13)$	$(9,15)$	$(11,18)$	$(12,20)$	$\cdots$

Figure 8: Wythoff pairs defined inductively.

Thus, every positive integer appears exactly once as a member of a pair, and exactly once as a difference. In particular, it follows that if a player A leaves a Wythoff pair, player B cannot move to a Wythoff pair. It is a little harder to see that if player B leaves a pair which is not one of the Wythoff pairs, player A can move to a Wythoff pair. Suppose player B leaves the pair (a, b) $(a \leq b)$ which is not one of the Wythoff pairs. If $a = b$, player A takes it all and wins the game. If not, let (a, a') or (a', a) be the Wythoff pair to which a belongs. If $a' < b$, player A reduces the larger b pile to a pile with a' counters. If $b < a'$ (so that $a < b < a'$ and $b - a < a' - a$), player A removes an equal number of counters from both piles, so as to leave the Wythoff pair whose difference is $b - a$. Thus player A can win, no matter what movements player B makes, unless player A finds a Wythoff pair before his first move (in which case he will concede, if he thinks player B knows how to play).

Even without a general formula, it is easy to exhibit a lot of Wythoff pairs (a_n, b_n), using the above recurrence. But in 1907, Wythoff discovered that the sequences $\{a_n\}$ and $\{b_n\}$ are two beautiful sequences that can be defined as:

$$a_n = [n\phi] \quad \text{and} \quad b_n = [n\phi^2],$$

where $\phi = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio. According to Harold S. M. Coxeter, (Coxeter, 1953), *such a formula was given by Wythoff "out of a hat"*, but a more usual approach can be provided with a beautiful theorem of Samuel Beatty, first proposed as a problem in the American Mathematical Monthly in 1926 (Beatty, 1926).

Winning Wythoff's Nim with Beatty

The sequences $\{a_n\}$ and $\{b_n\}$ are an example of *Beatty sequences*, so-named after Beatty's beautiful theorem:

If α, β are two positive irrational numbers satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

then the sequences

$$[\alpha], [2\alpha], [3\alpha], [4\alpha], \cdots \qquad \text{and} \qquad [\beta], [2\beta], [3\beta], [4\beta], \cdots$$

together include every positive integer just once.

The proof of Beatty's theorem was conceived jointly by J. Hyslop in Glasgow and A. Ostrowski in Göttingen and can be found in (Coxeter, 1953), (Mathews, 2004), (Ostrowski & Hyslop, 1927). For a given integer N, consider how many members there are of each sequence less than N. It is clear that there are $[N/\alpha]$ in the first sequence, and $[N/\beta]$ in the second. Therefore, there are

$$[N/\alpha] + [N/\beta]$$

members in the two sequences less than N. Since $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, where α and β are irrationals, N/α and N/β are two irrational numbers whose sum is the integer N. Hence their fractional parts must add up to exactly 1, and

$$[N/\alpha] + [N/\beta] = N - 1.$$

So there are $N - 1$ numbers in the two sequences less than N. Thus there is 1 number in the two sequences less than 2; it must be 1. Then there are 2 numbers present less than 3; they must be 1 and 2. Proceeding this way, it follows that every positive integer occurs exactly once in the two sequences. This is one of the requirements for the Wythoff pairs in Wythoff's Nim. The other, that the difference shall be n, is guaranteed by taking

$$\beta = \alpha + 1.$$

Since, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, it follows that

$$\alpha^2 - \alpha - 1 = 0,$$

whence $\alpha = \phi$, $\beta = \phi^2$, and the nth Wythoff pair is

$$([n\phi], [n\phi^2]).$$

We have just shown how to win Wythoff's Nim with Beatty:

Theorem (Winning Wythoff's Nim with Beatty): *The winning pairs of Wythoff's Nim are* (a_n, b_n) *and* (b_n, a_n) *for* $n \geq 0$*, where* a_n *and* b_n *are given as above.*

The algebraic characterization of the winning pairs of Wythoff's Nim, given by the last theorem, is computationally more efficient than the recursive characterization, but yet involves computation with irrational quantities. In contrast, in Bouton's Nim it is possible to characterize the winning positions using only the cardinality of the constituent piles by means of their binary representations. Thus we may ask: is there a more efficient method of computing moves for Wythoff's Nim that, in particular, places the two games on an equal footing with respect to the computation of play? Quoting Robert Silber, (Silber, 1976), *the connection of the Wythoff pairs with the golden ratio suggests to any "Fibonaccist" that the Fibonacci numbers are not very far out of the picture.* During the 1970's, investigations into Fibonacci representations revealed that the winning pairs for Wythoff's Nim are quite fundamental to the analysis of the Fibonacci number system. As a consequence of these investigations, Silber (Silber, 1977) presented a method of computing moves for Wythoff's Nim using Fibonacci representations. In the rest of this section we shall describe how Fibonacci representations mimic the role of binary representations in Bouton's Nim in determining the winning strategy for Wythoff's Nim. Furthermore, in the meantime, we shall enjoy some beautiful properties of the sequence $\{a_n\}$.

Winning Wythoff's Nim with Fibonacci

Let's return to the increasing Beatty sequence $\{a_n\}$:

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 24, 25, \cdots$$

The increments that occur in $\{a_n\}$ are either 1 or 2. If we look carefully to these increments (see Figure 9), we can realize that we have an increment of 2 in positions numbered $1, 3, 4, 6, 8, 9, 11, \ldots$, i.e., in the positions that occur in the sequence $\{a_n\}$.

1		3		4		6		8		9		11		12		14		16		17		19
	2		1		**2**		**2**		1		**2**		1		**2**		**2**		1		**2**	

Figure 9: Increments of 2 occur in positions that occur in the sequence $\{a_n\}$.

Hence, when n occurs in the sequence, $a_{n+1} = a_n + 2$; and if not, $a_{n+1} = a_n + 1$. This gives us another way to define recursively the sequence $\{a_n\}$, starting from $a_1 = 1$. Then, clearly, our sequence $\{b_n\}$ can be defined as the increasing sequence of all positive integers not in $\{a_n\}$ (more recurrence relations involving these two sequences can be found in (Mathews, 2004)). Thus, we have just generated again the Wythoff pairs. Among the Wythoff pairs we can find pairs of consecutive Fibonacci numbers: $(1, 2), (3, 5), (8, 13), (21, 34), \ldots$ (see Figure 10):

n	**1**	**2**	3	4	**5**	6	7	8	9	10	11	12	**13**	14	15	16
a_n	**1**	**3**	4	6	**8**	9	11	12	14	16	17	19	**21**	22	24	25
b_n	**2**	**5**	7	10	**13**	15	18	20	23	26	28	31	**34**	36	39	41

Figure 10: Wythoff pairs of consecutive Fibonacci numbers.

These pairs of consecutive Fibonacci numbers played a fundamental role in the analysis of the Fibonacci number system. We shall review some basic facts about this system, following the approach in (Silber, 1977). We shall adopt the convention that the first Fibonacci numbers are 1, 1, 2, ...

A *Fibonacci representation* consists of a finite sequence of zeros and ones, read positionally from right to left. A one in the nth position denotes the presence of the nth Fibonacci number. The number represented is determined by summing the Fibonacci number whose presence is indicated by a one. Thus, for example, 10000000 denotes $F_8 = 21$, 1100000 denotes $F_7 + F_6 = 21$ and 1010101 denotes $F_7 + F_5 + F_3 + F_1 = 21$. Clearly, Fibonacci representation is not unique.

A Fibonacci representation is said to be *canonical* if:

1. the representation contains no adjacent ones and
2. F_1 is not present (meaning that there is no 1 in the first position).

Zeckendorf's theorem (see, for instance, (Brown, 1964)) essentially states that the canonical form turns out to exist and to be unique for each positive integer.

A Fibonacci representation is said to be *second canonical* if:

1. the representation contains no adjacent ones and
2. the right-most one of the representation is in an odded numbered position.

The second canonical representation exists and is unique for each positive integer. Indeed, the second canonical representation of any positive integer n can be obtained by adding 1 'in Fibonacci' (meaning that any pair of adjacent ones is rounded up to a single one in the position immediately to the left of the pair) to the canonical representation of $n-1$. For instance, in the case of 21, 10000000 is the canonical representation and 1010101 is the second canonical representation. In what follows, we shall see that Fibonacci representations can be used to give the winning strategy for Wythoff's Nim. Let n be a positive integer which is represented canonically (not necessarily second canonically) in Fibonacci. The positive integer n is said to be an *A-number* if the right-most one in the representation of n occurs in an even-numbered position; otherwise it is called a *B-number*. Obviously, every positive integer is either an A-number or a B-number, but never both. We represent canonically in Fibonacci, in Figure 11, the first Wythoff pairs.

n	1	2	3	4	5	6	7	8
n	1	100	101	1001	10000	10001	10100	10101
a_n	1	3	4	6	8	9	11	12
a_n	10	1000	1010	10010	100000	100010	101000	101010
b_n	2	5	7	10	13	15	18	20
b_n	100	10000	10100	100100	1000000	1000100	1010000	1010100

Figure 11: Wythoff pairs represented canonically in Fibonacci.

We can see in Figure 11 that, in each Wythoff pair (a_n, b_n) represented, the smaller number a_n is an A-number, and that the canonical Fibonacci representation of b_n is equal to that of a_n with a zero adjoined at the right end (in other words, b_n is the quantity represented by the *left shift* of the representation of a_n. For future convenience, we introduce the *right shift* which is the representation obtained by deleting the digit at the right end).

Furthermore, if we represent each n, as in Figure 11, using the second canonical Fibonacci representation, the numbers a_n and b_n are then obtained from successive left shifts of the second canonical representation of n (clearly, the left shift of any second canonical representation is the canonical representation of an A-number).

In 1976, Silber (Silber, 1977) characterized the winning pairs of Wythoff's Nim (for convenience, the pair $(0, 0)$ shall be considered a Wythoff pair) and presented a specific algorithm for playing Wythoff's Nim.

Theorem (Winning Wythoff's Nim with Fibonacci): *The winning pairs of Wythoff's Nim are the pairs (a, b) and (b, a) that satisfy the following two conditions:*

1. *the smaller number a is an A-number, and*

2. *the canonical Fibonacci representation of b is equal to that of a with a zero adjoined at the right end.*

Algorithm for playing Wythoff's Nim:

0. Given a pair (a, b) represent a and b canonically in Fibonacci. If (a, b) is a winning pair, leave the game (if you think your opponent knows how to play). Otherwise, go to 1.

1. If the smaller number a of the pair is a B-number, determine the quantity $\overline{a}$ represented by the right shift of the canonical Fibonacci representation of a. Then reduce the larger number b to $\overline{a}$. We shall give an example. Suppose $(a, b) = (7, 13)$. Then in canonical form, $a = 10100$ and $b = 1000000$. Hence a is a B-number and so we reduce b to $\overline{a} = 1010$. We obtain the winning pair $(\overline{a}, a) = (4, 7)$.

2. If the smaller number a of the pair is an A-number, and if in addition the larger number b of the pair exceeds the quantity $\underline{a}$ represented by the left shift of the canonical representation of a, reduce b to $\underline{a}$. We shall illustrate this situation. Suppose $(a, b) = (11, 20)$. Then in canonical form, $a = 101000$ and $b = 1010100$. Hence a is an A-number and $\underline{a} = 1010000$ which is less than b. So we reduce b to $\underline{a}$. We obtain the winning pair $(a, \underline{a}) = (11, 18)$.

3. If neither 0, 1 nor 2 holds, determine the second canonical Fibonacci representation of the positive difference of the members of the pair. A left shift on this representation will produce an A-number and another left shift will produce a B-number. Together they form a winning pair that can be obtained from (a, b) by the reduction of a and b by an equal amount. We shall give an example. Suppose $(a, b) = (25, 33)$. Then in canonical form, $a = 10001010$ and $b = 10101010$. We have that a is an A-number with $\underline{a} = 100010100 > b$. Hence we compute $b - a = 8$. The second canonical representation of 8 is 10101. Thus, the canonical representation of a_8 is 101010 and that of b_8 is 1010100. We obtain the winning pair $(12, 20)$, reducing both 25 and 33 by 13.

We shall end this paper with the Beatty sequence $\{a_n\}$.

We have already noticed that among the Wythoff pairs we can find pairs of consecutive Fibonacci numbers:

$$(1, 2),\ (3, 5),\ (8, 13),\ \cdots .$$

Clearly, there are many Wythoff pairs whose members are not Fibonacci numbers; the first such pair is $(4, 7)$. The Wythoff pair $(4, 7)$ can be used to generate a Fibonacci sequence in the same way that the Wythoff pair $(1, 2)$ can be taken to determine the usual Fibonacci numbers. If we apply the Fibonacci recurrence to 4 and 7 as starting values, we obtain the pairs:

$$(4, 7),\ (11, 18),\ (29, 47),\ \cdots .$$

It is perhaps a little surprising that all these pairs are Wythoff pairs! (see Figure 12). Then if we start with the Wythoff pair $(6, 10)$ and repete the Fibonacci recurrence we obtain

$$(6, 10),\ (16, 26),\ (42, 68),\ \cdots,$$

all again Wythoff pairs! In general, for any Wythoff pair, the Fibonacci sequence starting in that pair occurs completely in Wythoff pairs and it is very easy to discover the next Wythoff pair in the sequence. Just note that, given any Wythoff pair (a_n, b_n), we have that $(a_n+b_n, a_n+2b_n) = (a_{b_n}, b_{b_n})$ (see, for instance, (Mathews, 2004)).

n	**1**	(2)	**3**	**4**	(5)	**6**	(7)	**8**	**9**	(10)	**11**	**12**	(13)	**14**
a_n	$\underline{1}$	$\underline{3}$	$\underline{\underline{4}}$	$\underline{\underline{\underline{6}}}$	$\underline{8}$	9	$\underline{\underline{11}}$	12	14	$\underline{\underline{\underline{16}}}$	17	19	$\underline{21}$	22
b_n	$\underline{2}$	$\underline{5}$	$\underline{\underline{7}}$	$\underline{\underline{\underline{10}}}$	$\underline{13}$	15	$\underline{\underline{18}}$	20	23	$\underline{\underline{\underline{26}}}$	28	31	$\underline{34}$	36

Figure 12: Wythoff pairs generate Wythoff pairs under the Fibonacci recurrence.

In the Fibonacci number system it is even easier to reach the same conclusion. Let (a_n, b_n) $(a_n < b_n)$ be a Wythoff pair. Then the smaller number a_n is an A-number and b_n is the left shift of the canonical representation of a_n. It follows immediately from the Fibonacci number system that $a_n + b_n$ has a representation which is exactly the left shift of b_n and $(a_n + b_n) + b_n$ has a representation which is exactly the left shift of $a_n + b_n$. *Hence, the Fibonacci sequence generated by any Wythoff pair, when represented canonically in the Fibonacci number system, consists of consecutive left shifts of the first term of the sequence* (Silber, 1976). We give the simplest example and generate (see Figure 13), in the Fibonacci number system, the classical Fibonacci sequence with starting values 1 and 2:

1	2	3	5	8	$\cdots$
10	100	1000	10000	100000	$\cdots$

Figure 13: Fibonacci sequence expressed in the Fibonacci number system.

A Wythoff pair is said to be *primitive* (see (Silber, 1976)) if no other Wythoff pair generates it. If we look carefully to Figure 12, we can see that the first primitive pairs occur in positions numbered $1, 3, 4, 6, 8, 11, 12, 14, \ldots$, i.e., in the positions that occur in the sequence $\{a_n\}$. In fact, it is easy to show (see (Silber, 1976)) that a Wythoff pair (a_n, b_n) is primitive if and only if $n = a_k$ for some positive integer k, i.e., the primitive pairs correspond exactly to the cases in which n occurs in our sequence $\{a_n\}$. Hence, the number of primitive pairs is infinite. It follows immediately that there exists a sequence of Fibonacci sequences which simply covers the set of positive integers. It is also easy to realize (see (Silber, 1976)) that the members of our beautiful sequence $\{a_n\}$ are precisely those numbers whose second canonical Fibonacci representation contain a 1 in the first position. These beautiful properties of sequence $\{a_n\}$ can be seen in Figure 14:

n	**1**	2	**3**	**4**	5	**6**	7	**8**
n	**1**	100	**101**	**1001**	10000	**100001**	10100	**10101**
a_n	1	3	4	6	8	9	11	12
a_n	**10**	1000	**1010**	**10010**	100000	**1000010**	101000	**101010**
b_n	2	5	7	10	13	15	18	20
b_n	**100**	10000	**10100**	**100100**	1000000	**10000100**	1010000	**1010100**

Figure 14: Beautiful properties of sequence $\{a_n\}$.

References

Beatty, Samuel (1926). Problem 3173. *Amer. Math. Monthly* **33**, 159.

Berlekamp, Elwyn R., Conway, John H., & Guy, Richard K. (2001). *Winning Ways.* A K Peters, Ltd., Natick, Massachusetts, 2nd edition. First edition published in 1982 by Academic Press.

Berge, Claude (1962). *The Theory of Graphs and its Applications.* London, Methuen; New York, Wiley. Translated by Alison Doig.

Bouton, Charles L. (1901-1902). Nim, a game with a complete mathematical theory. *Ann. of Math.*, 2nd Ser., **3**, 35-39.

Brown, J. L. (1964). Jr. Zeckendorf's theorem and some applications. *Fibonacci Quart.* **2**, 163-168.

Coxeter, H. S. M. (1953). The golden section, phyllotaxis, and Wythoff's game. *Scripta Math.* **19**, 135-143.

Gardner, Martin (1988). *Hexaflexagons and other mathematical diversions.* The University of Chicago Press.

Gardner, Martin (1997). Wythoff's Nim, in: *Penrose Tiles to Trapdoor Ciphers.* The Mathematical Association of America, Washington.

Grundy, Patrick M. (1939). Mathematics and games. *Eureka* **2**, 6-8.

Mathews, Daniel (2004). A beautiful sequence. *Aust. M. S. Gazette* **31**(1), 12-17.

Neto, João P., & Silva, Jorge N. (2004). *Jogos Matemáticos, Jogos Abstractos.* Gradiva.

Nowakowski, Richard J. (2008). History of Combinatorial Game Theory. *Proceedings of the Board Game Studies Colloquium XI*, Lisbon.

Ostrowski, A., & Hyslop, J. (1927). Solution to Problem 3177. *Amer. Math. Monthly* **34**, 159.

Santos, Carlos P. (2010). *Some notes on impartial games and NIM dimension.* PhD thesis, Faculdade de Ciências, Universidade de Lisboa.

Silber, Robert (1976). A Fibonacci property of Wythoff pairs. *Fibonacci Quart.* **14**, 380-384.

Silber, Robert (1977). Wythoff's Nim and Fibonacci Representations. *Fibonacci Quart.* **15**, 85-88.

Sprague, Roland (1935-36). Über mathematische kampfspiele. *Tôhoku Math. J.* **41**, 438-444.

Wythoff, Willem A. (1907). A modification of the game of Nim. *Nieuw Arch. Wisk.* **7**, 199-202.

Some medieval problems

Joaquim Nogueira

In the eighth century lived the Saxon scholar Ealhwine, or Alchvine, usually known by Alcuin of York or by his Latin name Albinus (b. 735 in Northumbria). Alcuin played an essential part in the so-called «Carolingian Renaissance» founding the palace school at Aachen where the seven liberal arts were taught according to the educational system of Cassiodorus (c. 490-580). His most important writings were his revisions of the *Vulgata* and his voluminous letters, the latter being collated in the 9$^{\text{th}}$ century as a model of Latin composition. He wrote several basic texts concerning arithmetic, geometry and astronomy, something that was no easy task at a time when scientific knowledge was scarce and usually restricted to monasteries. Alcuin took up the position of abbot at the abbey of St-Martin of Tours (France), where he founded an important library and school, and where he remained until his death on the 19th of May, 804.

He was also close friend of Charlemagne, emperor of the Francs. According to Einhard (a biographer of Charlemagne) Alcuin «was the most educated man one could ever find»; Notker (another biographer of the emperor) wrote that «he was more skilled in all fields of knowledge than anyone else…» The emperor, well aware of these qualities, considered himself the pupil of Alcuin and treated Alcuin as his master.

Alcuin of York (735-804).

It is believed that, in the year 799, Alcuin sent to Charlemagne a collection of 53 (some authors say 56) recreational mathematics problems, «…some arithmetic curiosities to entertain himself…», in his leisure time. Nowadays it is commonly believed that this «masterpiece» was the text «Propositiones ad acuendos juvenes» («Problems to help develop the minds of youngsters»). In the first colloquium, two years ago, we related the history of the puzzles in which one must cross the river (Problems 17, 18 and 19 of the «Propositiones»), along with some of its modern developments. In the following I will analyse some of the other problems Alcuin collected. We start with Problem 42:

«XLII. PROPOSITIO DE SCALA HABENTE GRADVS CENTVM. Est scala una habens gradus C. In primo gradu sedebat columba una: in secundo duae; in tertio tres; in quarto IIII; in quinto V. Sic in omni gradu usque ad centesimum. Dicat, qui potest, quot columbae in totum fuerunt?»

A translation follows:

«A ladder has 100 steps. A pigeon landed on the first step, two pigeons landed on the second step, three on the third, four on the fourth, and so on, up to the hundredth step. How many pigeons had landed on the ladder?»

The solution is the following: on the first and hundredth steps one finds 101 pigeons; on the second and ninety-ninth, one again finds 101 pigeons, and so on until we get to the fiftieth and fifty-first steps were, again, one finds 101 pigeons. As there exist fifty such pairings, the sum is $50 \times 101 = 5050$.

We may notice that Alcuin had a slightly different solution: he remarked that the pigeons on the first and ninety-ninth steps were 100, as so were the pigeons on the second and ninety-eight, on the third and ninety-seventh, and so on. As he didn't count the fiftieth and hundredth steps, there were 49 such pairings which summed up to a total of 4900. After that he summed the 150 pigeons that were on the remaining steps, obtaining again the result of 5050.

According to Wolfgang Sartorius, Baron von Waltershausen, the remarkable mathematician Carl Friedrich Gauss (1777-1855) was an unexpected protagonist in a similar story when he was still attending basic school: his teacher (J.G. Büttner, the director of St. Katharine's Volksschule) asked the class to compute the sum of an arithmetic progression; later tradition established that this was the sequence of all natural numbers from 1 and 100. While his colleagues were laboriously summing the numbers, and occasionally making mistakes, the young Gauss didn't take long with this task because he noticed that the numbers could be paired as described in the first solution above. According to Brian Hayes: «The problem was barely stated before Gauss threw his slate on the table with the words: *There it lies.* While the other pupils continued [counting, multiplying and adding], Büttner, with conscious dignity, walked back and forth, occasionally throwing an ironical, pitying glance towards the youngest of the pupils. The boy sat quietly with his task ended, as fully aware as he always was on finishing a task that the problem had been correctly solved and that there could be no other result. At the end of the hour the slates were turned bottom up. That of the young Gauss with one solitary figure lay on top. When Büttner read out the answer, to the surprise of all present that of young Gauss was found to be correct, whereas many of the others were wrong.»

According to Eric Temple Bell (who had a reputation of being a highly inventive writer), the problem proposed by Gauss's teacher was, in fact, much more difficult because the sequence of numbers that his pupils were supposed to sum began with 81.297, 81.495 and ended in 100.701, 100.899, where the difference between two consecutive numbers was 198. The solution is the following:

$$\begin{aligned}
&81297 + 81495 + \ldots + 100701 + 100899 = \\
&81297 + (81297 + 198) + \ldots + (81297 + 98 \times 198) + (81297 + 198 \times 99) = \\
&100 \times 81297 + (0 + 1 + 2 + \ldots + 98 + 99) \times 198 = \\
&8.129.700 + (50 \times 99) \times 198 = 9.199.800
\end{aligned}$$

Therefore the sum is 9.199.800.

Gauss, in his classroom, summing the first hundred positive integers.

We may also notice that Brian Hayes makes a very thorough review of the several appearances of this anecdote in literature, in «Gauss's Day of Reckoning», published in *American Scientist* of May-June 2006, and that the formula that gives the sum, $S(n)$, of all natural numbers from 1 to n, $S(n) = \dfrac{n(n+1)}{2}$, was generalized in the early 17^{th} century, by Johann Faulhaber, to a formula that (making use of Bernoulli numbers) sums the k^{th}-powers of the numbers from 1 to n.

Now we set aside the sums and consider a puzzle involving quotients. Problem 43 of Alcuin's «Propositiones» is the following: «Homo quidam habuit CCC porcos, et iussit, ut tot porci numero impari in III dies occidi deberent. Similis est et de XXX sententia. Dicat, qui potest, quot porci impares siue de CCC siue de XXX, inter tres dies [ter] occidendi sunt?», which translates to «A certain man has 300 pigs. He ordered all of them slaughtered in 3 days, but with an odd number killed each day. What number were to be killed each day?»

It is obvious that this problem has no solution because three odd numbers can never add up to an even number. Alcuin himself was aware of it; he clearly stated this fact sarcastically adding that this problem was left for the children. Most probably this problem seems to be included for occupying troublesome students.

It is interesting to remark that the english language allows the existence of a comical solution. One kills one pig the first day, a second pig the second day and leaves the remaining 298 pigs to be turned into bacon on the third day. But wait! The number 298 is not odd! That's true but even so one has a solution because «298 is a very odd number of pigs to kill in one day», making a pun with the double meaning of the word «odd» («strange» and «not divisible by two»).

A slightly different version of the problem in which the killing of the pigs is not required but instead its confining in three fences, such that each one must contain an odd number of swine. This version is solvable although using something that looks like cheating. It is enough to confine 297 pigs in a fence, the other 3 in another fence and inside it another fence confining one single pig. The well-known american puzzler Sam Loyd, in his Cyclopedia, poses a similar problem, «The Pig Sty», with a similar solution.

The Pig Sty.

We add another comical version: the newspaper *The Royal Magazine*, of November 1911 challenges its readers by asking the question: «A farmer owns 17 donkeys. One of his friends bets with him that he is not able to put them inside four fences such that in each fence there is an odd number of donkeys. But the farmer managed to solve the problem. How?»

The farmer began by putting inside each fence, respectively, 7, 5, 3, 2 beasts. After inspecting the first three fences his friend claimed victory, but the farmer told him to get inside the last fence and check how many beasts were there. As soon as he got inside the fence the farmer locked him in, claiming that inside the fence there were three donkeys.

Playing with numbers W. Leybourn, in his book «Pleasure with Profit» (1694), asks how is it possible to to add five odd numbers so that its sum is 20. The answer is really simple: one adds two *ones* with three *nines* and read the numbers upside down. And, in fact

$$\text{«}\,|+|+9+9+9\text{»} = 20.$$

In the book «Illustrated book of Puzzles», of Don Lemon (1890) the author plays with numbers:

Twice six are eight of us,

Six are but three of us,

Nine are but four of us,

What can we possibly be?

Would you know more of us?

I'll tell you more of us,

Twelve are but six of us,

Five are but four of us,

now do you see?

Let's get back to Alcuin: Problem 41 of the «Propositiones ad acuendos juvenes» is also very interesting:

«XLI. PROPOSITIO DE SODE ET SCROFA. Quidam Paterfamilias stabiliuit curtem nouam [quadrangulam], in qua posuit scrofam, quae peperit porcellos VII in media sode, qui una cum matre, quae octaua est [F. add., octo sunt], pepererunt igitur unusquisque in omni angulo VII. Et ipsa iterum in media sode cum omnibus generatis peperit VII. Dicat, qui uult, una cum matribus quot porci fuerunt?», which translates to:

The mother pig and some of its daughters.

«A certain head of household set up a new [quadrangular] enclosure in which he placed a sow. The sow gave birth to seven piglets in the middle of the sty. The offspring, along with the mother, the eighth pig, each gave birth to another seven piglets in each corner [of the sty]. Then, in the middle of the sty, the mother and all her offspring [each] gave birth to seven more. How many pigs were there [in the end], including the mother?»

It is clear that the answer one wishes is not «Many!». Think a little: in the beginning we have the mother and the seven daughters ($1+7=8$). Later these pigs gave birth to seven pigs, each one ($(1+7)\times 7=56$). Finally both the mother, daughters and grand-daughters gave birth, again, to seven piglets ($(1+7+56)\times 7=64\times 7=448$). Thus, assuming that none have died there were at the end of this birth process $1+7+56+448=512$ pigs inside the fence[1].

The main interest in this problem comes from the fact that, four centuries before the rabbits of Leonardo Pisano (i.e., Fibonacci) (c.1170-c.1250), one finds a mathematical problem in which several immortal beasts are confined inside a fence, gave birth several times, each time the number of new-born is the same, and one wishes to find the total number of beasts inside the fence after a certain time period; in its essence it is a problem similar to the one suggested by Fibonacci in his «Liber Abaci» and that, in more recent times, gave rise to the mathematical theory of linear recurrences. Perhaps some of the main inspirational sources of Leonardo were the problems suggested by Alcuin in his «Propositiones ad acuendos juvenes».

Let's return back to Alcuin of York, once again, to study problem 51: «LII. PROPOSITIO DE HOMINE PATERFAMILIAS. Quidam paterfamilias iussit XC modia frumenti de una domo sua ad alteram deportari; quae distabat leucas XXX: ea uero ratione, ut uno camelo totum illud frumentum deportaretur in tribus subuectionibus, et in unaquaque subuectione XXX modia portarentur: camelus quoque in unaquaque leuca comedat modium unum. Dicat, qui uelit, quot modii residui fuissent?»

The translation is the following: «A certain head of household ordered that 90 modia of grain be taken from one of his houses to another 30 leagues away. Given that this load of grain can be carried by a camel in three trips, and that [the camel] eats one modium per league, how many modia were left over [at the end of the transport]?»

Although not clearly stated it is assumed that the camel can carry at most 30 modia of grain. According to Alcuin the answer to the problem is given by 20 modia of grain because, he reasoned, a camel does not eat when is not carrying anything (that is, when it returns home). In the first trip it carries 30 modia of grain, travels 20 leagues eating 20 modia and leaves the remaining 10 on the ground; then it returns home. In the second trip it does exactly the same. In the third trip, after traveling 20 leagues (and consuming 20 modia of grain), there is room left to carry the 20 modia that were on the ground. After eating 10 modia of grain it reaches its goal carrying with the remaining 20 modia.

A better solution occurs if the camel carries 30 modia of grain, travels 10 leagues eating 10 modia and leaves the remaining 20 on the ground; then it returns home. In a second trip it carries 30 modia of grain, loads 10 after traveling 10 leagues, travels 15 leagues more where it unloads the 15 modia remaining; it returns home. In a third trip it carries 30 modia of grain and after traveling 10 leagues loads the remaining 10 that where on the ground. After traveling 15 leagues more (and eating 15 modia) it loads the 15 that were on the ground. After eating 5 modia it arrives to destination with 25.

Notice that if the camel eats on its way back home, five trips are needed (in both directions) in order to carry only... 10 modia of grain! On the first trip it carries 30 modia, leaves 10 on the

[1] Alcuin's Problem 49, «Seven carpenters [each] made seven wheels. How many carts did they build?» looks like an easier version of problem 41. One possible answer is «None» because the carpenters were too busy making wheels. Another answer, assuming that a cart has four wheels, is 12 with one spare wheel.

ground (at the 10th league) and returns home. In the second trip, it carries again 30 modia of grain. When it reaches the 10^{th} league (and after eating 10 modia of grain) there is room to carry the 10 modia that were left on the ground, on the first trip. When it reaches league 15 it leaves 10 modia on the ground and gets back home. Finally, on the third trip, it carries 30 modia, spends 15 until it gets to the 15^{th} league. There it picks up the 10 modia that had been left on the ground, thus carrying 25. After another 15 leagues, in which 15 modia are eaten, it reaches its goal with 10 modia of grain.

The hero of the puzzle (older versions).

And if the camel was strong enough to carry 45 modia? Then the answer would be the following: in the first trip the camel carries 45 modia, stops after 15 leagues (and after eating 15 modia of grain), leaves 15 on the ground and returns eating the remaining 15; this is the second trip. Then it loads the 45 modia that were left back home and begins the third trip. When arriving half-way it had already eaten 15 modia. The 15 that were left on the ground, in the first trip, are now loaded: he is carrying again 45. Until the camel reaches its destination 15 more modia are eaten, which means that from the 90 modia with which it began, only 30 got to destination.

During the last millennium several authors repeatedly presented similar problems: Luca Pacioli in his «De Viribus» shows how to carry the highest number of apples between two towns (and not eating everything when traveling), how ships minimize the tax they pay at consecutive customs posts, or how to carry 100 pearls 10 miles, 10 at a time, leaving one every mile (but not when returning home). The solution that he gives to the latter problem is the following: one takes them 2 miles in ten trips, leaving 80 on the ground. Then one takes them to destination in 8 trips, getting 16 to the destination. Note that this solution can easily be improved: if one travels a miles ($0 \leq a \leq 10$) and make a single stop the number of pearls that arrive to destination is given by

$$\begin{aligned}(100-10a)-(10-a)^2 &= 10(10-a)-(10-a)^2 = \\ (10-(10-a))(10-a) &= a(10-a).\end{aligned}$$

The maximum of this function is attained when $a = 5$ and is given by 25 pearls.

Note that it is possible to make more stops, but this is restricted by the fact that pearls cannot be divided. Citing David Singmaster:

«...if the amount of pearls accumulated at each depot is a multiple of ten, one can get 28 to the destination by using depots at 2 and 7 or 5 and 7. One can get 27 to the destination with depots at 4 and 9 or 5 and 9.

If the material being transported was a continuous material like grain, then the optimal method seems to firstly move 1 mile to get 90 there, then move another $\dfrac{10}{9}$ to get 80 there, then another $\dfrac{10}{8}$ to get 70 there, ..., continuing until we get 40 at $8,4563...$, and then make four trips to the destination. This gets $33,8254$ to the destination.»

Cardano (1539), Mittenzwey (1880), Pearson (1907), Sam Loyd (1910) and many other mathematicians in more recent times have also challenged their readers with problems in the same vein.

The modern versions of problem 52 of Alcuin are usually known as the «jeep problems» because most of them cover situations in which a jeep is crossing the desert carrying several gas cans. Usually one wants to find out how many miles a jeep can travel with the gas it carries; it is allowed to leave, somewhere, some of the gas carried, and to return back to replenish (in full or in part) its tank. Back home one usually supposes that there is no gas shortage.

Suppose that the jeep only carries 1 gas can, which is enough to completely fill its tank; with it the jeep is able to to travel, say, one mile.

Now suppose that the jeep can carry two gas cans (but its tank remains the same). So that its traveling distance may be maximized he travels $\frac{1}{3}$ of a mile, stores $\frac{1}{3}$ of a gas can on the ground and gets back with the $\frac{1}{3}$ remaining. Its tank is refilled with the second gas can, then travels to the spot where $\frac{1}{3}$ of a can had been stored, refills its tank and travels one mile. That means that the jeep can travel $\frac{4}{3}$ of a mile.

The hero of the puzzle (modern versions).

The maximal solution for this problem for n fuel cans is given[2] by

$$1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}.$$

[2]G. G. Alway. Note 2707: Crossing the desert. MG 41 (No. 337) (1957) pp. 209, or Pyle, I. C. The explorer's problem. Eureka 21 (Oct 1958) pp. 5-7.

If n ranges from 1 to 5 this formula gives

$$1, \frac{4}{3}, \frac{23}{15}, \frac{176}{105}, \frac{563}{315}.$$

This solution is also given by Rouse Ball in his «Exploration problems» (1911) and in «Mathematical Recreations and Essays» (1^{st} edition 1892) in which he distinguishes several forms of the problem, and gives their solutions: suppose that n explorers, that can carry food for d days wish to travel the maximum they can into the desert in only one trip each (note that it is easy to prove that this is equivalent to have one single explorer doing n trips). Then:

($A1$) If depots are permitted they allow one man to travel

$$d(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n})$$

length units into the desert and returning back;

Rouse Ball, author of «Mathematical Recreations and Essays».

($A2$) if the explorers do not return back or alternatively the explorer does not eat when returning back it is possible to travel

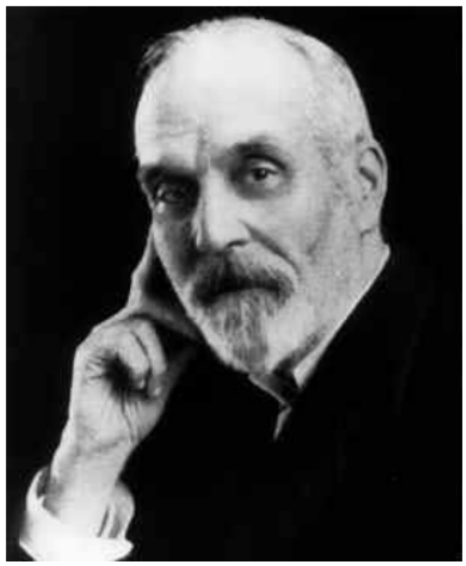
$$d(1 + \frac{1}{3} + \frac{1}{5} \ldots + \frac{1}{2n-1})$$

length units into the desert. The problems referred to above as «jeep problems», are an example of this situation.

(B) if depots are not allowed it is possible that one single man travels for $\frac{nd}{n+1}$ days into the desert and returning back.

A variation of the problem, due to Henry Dudeney, is the following: «Nine travelers, each one with his own car, are at the eastern edge of a desert. Their goal is to explore the desert going west, without returning to refill their tanks. Each car can travel forty kilometers with a full tank, whose capacity is 5 liters. They can also carry in their trunks, at most, 9 cans of 5 liters and cans can be exchanged between cars. Noting that each car must keep enough gas to return home safely, what is the maximum distance they can travel into the desert?»

The solution is the following: the nine explorers travel together the first 40 kilometers. Then one of them gives all the others a gas can, refills his own tank (that was empty) with the remaining gas can and gets back home. His teammates also fill their own tanks with the can they received and travel 40 kilometers more. Then one of the travelers gives to each of his companions a gas can, keeping two so that he can return home. This procedure is repeated several times: the last explorer after traveling $9 \times 40 = 360$ kilometers has still in his trunk 9 cans of 5 liters which allow him to return safely back home.



Henry Dudeney, author of «Amusements in Mathematics».

Note that this is essentially problem (B) with $n = 9$ (nine explorers) and $d = 10$ (because each car carries its own tank plus 9 cans of fuel). So one of the cars is able to travel for

$$\frac{nd}{n+1} = \frac{9 \times 10}{10} = 9$$

days. As in each day it travels 40 kilometers, after 9 days it travels 360 kilometers.

Brian Bolt proposes the following version: «A truck loaded with fuel and supplies can cross 400 kilometers through the desert. At the starting point, located on the edge of the desert, there is an almost endless supply of both products. Strategically placing some supplies on the route the truck is supposed to follow, it is expected that it could travel across the desert and return back home traveling a lot more than the 200 kilometers allowed by one single filling of its tank. From the starting point how many trips would be needed to travel 600 kilometers into the desert and safely return?»

It is clear that a single trip allows the truck to travel only 200 kilometers.

And with two trips? Suppose that, on the first trip, after traveling 100 kilometers, it stores on the ground enough supplies to travel 200 kilometers and then gets back. On the second trip, when reaching 100 kilometers, because $\frac{1}{4}$ of its supplies are already spent, it replaces them by picking up half of the supplies left on the previous trip. It can go another 200 kilometers into the desert and returns safely to the 100 kilometers spot. At this point it has exhausted its supplies but this is not a problem because it can load the remaining supplies on the ground and return back to the starting point. Thus, with two trips the truck can reach 300 (that is, $200 \times (1 + \frac{1}{2})$) kilometers.

If the truck were to travel three times into the desert the process would be the following: after spending $\frac{1}{6}$ of the supplies it would stop (this stopping point will be called A) to unload $\frac{2}{3}$ of the supplies and would return back home with the $\frac{1}{6}$ remaining. In the second trip it spends $\frac{1}{6}$ of the supplies to reach point A where it replenishes what was spent from the supplies stored on the ground. It goes on until $\frac{1}{4}$ of its supplies are spent. It stops (this stopping point will be called B) to unload half of its maximum supplies and then returns to point A. There it loads again $\frac{1}{6}$ of the supplies and goes back to the starting point. Finally on the third trip it spends $\frac{1}{6}$ of the supplies to reach point A; there it loads $\frac{1}{6}$ of the supplies stored on the ground. Now it spends $\frac{1}{4}$ of its supplies until reaching point B; there it loads $\frac{1}{4}$ of supplies. Next it travels until it has spent $\frac{1}{2}$ of the supplies. Then it returns to the starting point refilling in B and A what had been left on the ground. Thus, with three trips, the truck can reach 366, 7 (that is $200 \times (1 + \frac{1}{2} + \frac{1}{3})$) kilometers.

From these examples it is now clear that if the truck is allowed to travel n times into the desert it can cross

$$200 \times (1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n})$$

kilometers. That means that, because the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

diverges it could, at least in theory, cross any required distance. When the distance is 600 kilometers the number of trips that are needed is eleven. This is similar to problem ($A1$) with the restriction that the number of gas cans the jeep can carry is constant.

The problem was completely solved by N. J. Fine (*The jeep problem*, 1947) and C. G. Phipps. (*The jeep problem: a more general solution*, 1947) in its classical versions. Recently some new variations have risen along with new algorithms specially adapted to them. As an example note that in 1996 Gunter Rote and Guochuan Zhang, in *Optimal Logistics for Expeditions– The Jeep Problem with Complete Refilling*, improving some results obtained by Derick Wood (1984) and Ute and Wilfried Bauer (1989), studied in detail the problem in which one has n cans of fuel on the edge of a desert, a jeep with an empty tank whose capacity is just 1 can, the jeep can carry one can in addition to the fuel in its tank and moreover, when a can is opened, the fuel must immediately be filled into the jeep's tank. This is rather different from the situations studied before in which a gas can stored at a depot can be used several times as long as it's not empty.

Many more variations can be considered: what if, at some point, a bonus arises (a can full of fuel is found somewhere along the path of the explorers, or an extra car that suddenly appears and helps the others) or if the fuel consumption of the jeep is not constant (due to the characteristics of the terrain where it travels, or depending of the load, that is, the vehicle would use less fuel when it had less load to carry).

Notice that this kind of problems can have a practical application in wartime situations, especially with respect to fuel efficiency. Citing *Wikipedia*: «In the context of the bombing of Japan in World War II by B-29s, Robert McNamara says in the film *The Fog of War* that understanding the fuel efficiency issue caused by having to transport the fuel to forward bases was the main reason why the strategy of launching bombing raids from mainland China was abandoned in favor of the island hopping strategy:

Some B-29 flying to Japan.

We had to fly those planes from the bases in Kansas to India. Then we had to fly fuel over the hump into China. [...] We were supposed to take these B-29s ? There were no tanker aircraft there. We were to fill them with fuel, fly from India to Chengtu; offload the fuel; fly back to India; make enough missions to build up fuel in Chengtu; fly to Yawata, Japan; bomb the steel mills; and go back to India.

We had so little training on this problem of maximizing [fuel] efficiency, we actually found to get some of the B-29s back instead of offloading fuel, they had to take it on. To make a long story short, it wasn't worth a damn. And it was LeMay who really came to that conclusion, and led the Chiefs to move the whole thing to the Marianas, which devastated Japan.»

References

Alway, G. G. (1958). *Note 2707: Crossing the desert.* MG 41 (N. 337) pp. 209.

Ball, R. (1892). *Mathematical Recreations and Essays*, 1st edition.

de Bondt, M. (1996). *The camel-banana problem*, Nieuw Archief voor Wiskunde, 14, pp. 415-426.

de Bondt, M. (2010). *Exploring Mount Neverest*, arXiv:1009.2938v1.

Brauer, U., & Brauer, W. (June, 1989). *A new approach to the jeep problem*, Bull. EATCS N. 38, pp.145-154.

Dewdney, A. K. (1987). *Computer Recreations*, Sci. Amer. 256, 6 (June 1987), pp. 106-109. Solutions in 257, 4 (October 1987), p. 169, and 257, 5 (November 1987), p. 122.

Dudeney, Henry E. (1917). *Amusements in Mathematics*, Thomas Nelson & Sons, 1917. Reprinted by Dover Publications, 1958.

Fine, N. J. (January, 1947). *The jeep problem*, Amer. Math. Monthly 54, pp. 24-31. M. Folkerts (1978).

Franklin, J. N. (September, 1960). *The range of a fleet of aircraft*, J. Soc. Indust. Appis. Math. Vol. 8, pp. 541-548, N3.

Gale, D. (1970). *The jeep once more or jeeper by the dozen*, Amer. Math. Monthly 77, pp. 493-501.

Gale, D. (1994). *The return of the jeep* , Math. Intelligencer 16, N. 1, Winter, pp. 42-44.

Hadley, J. & Singmaster, D. (March, 1992). *Problems to sharpen the young*, Math. Gazette 76 (N. 475), pp. 102-126.

Rote, G., & Zhang, G. (1996). *Optimal Logistics for Expeditions– The Jeep Problem with Complete Refilling*, Optmierung und Kontrolle 71 (Spezialforschungsbereich F 003) (Karl-Franzens-Universitat Graz & Technische Universitat Graz).

EXTREME ALPHAMETICS

Mike Keith

Introduction

An alphametic is a puzzle such as SEND + MORE = MONEY, composed by Dudeney in 1924 (Dudeney, 1924), in which letters are to be replaced with decimal digits so that the result is a correct addition sum. The unique solution to Dudeney's puzzle is S=9, E=5, N=6, D=7, M=1, O=0, R=8, Y=2, giving the sum 9567 + 1085 = 10652. Note that the letters-to-digits mapping is required to be one-to-one: a given letter always represents the same digit, and a given digit is always encoded by the same letter. In this paper we discuss the construction and solution of various kinds of "extreme" alphametics that push the boundaries of the form, as well as a problem that is closely related: solving an alphametic efficiently with a computer program.

Given a specific alphametic we are interested in its solvability. There are three possibilities:

1. It does not have a solution.
2. It has multiple solutions. For example, USE + LESS = SONNY (what the father might have written back after he received Dudeney's puzzle in the post from his son) has three solutions: 814 + 9411 = 10225, 715 + 9511 = 10226 and 517 + 9711 = 10228.
3. It has exactly one solution.

An alphametic of the third kind, with a unique solution, is called *pure*. In this paper, *all alphametics are required to be pure*. Note that we have implicitly assumed base 10 thus far. Although alphametic construction and solution is possible in any base, all results presented in this paper are for base 10.

Alphametic Solution by Computer

The first step to efficient algorithmic solution of an alphametic is to treat it not as a long-hand addition but as a Diophantine equation. From the properties of positional notation we know that SEND + MORE = MONEY actually means, in base 10,

$$(1000s + 100e + 10n + d) + (1000m + 100o + 10r + e) = (10000m + 1000o + 100n + 10e + y)$$

where we have used lowercase letters for the variables in order to avoid confusion between capital O and the digit 0. Rearranging this equation gives

$$9000m - 1000s + 900o - 91e + 90n - 10r + y - d = 0$$

To solve the alphametic we need to find the solution(s) to this linear Diophantine equation that satisfy two additional constraints:

1. Variable values must be integers in $[0, 9]$, except for s and m which must be in $[1, 9]$.
2. All variable values must be distinct.

The second part of condition (1) is due to the fact that s and m are both used as the first letter in a word (sEND, mORE, mONEY). By convention integers are never written with a leading zero, so any variable which appears as the first letter in a word cannot be assigned the value 0.

In general, we have a Diophantine equation in n variables (say, x_i), of the form

$$c_0x_0 + c_1x_1 + ... + c_{n-1}x_{n-1} = 0 \tag{1}$$

where the c_i are constant integer coefficients. An auxiliary array, f_i, has $f_i = 1$ if x_i is a variable which is not allowed to have the value 0 (otherwise, $f_i = 0$). A recursive algorithm for finding all solutions to the Diophantine equation (also obeying the two side constraints) follows from considering the situation where the first k variables have already been assigned digits. Then equation (1) becomes

$$c_kx_k + c_{k+1}x_{k+1} + ... + c_{n-1}x_{n-1} = rhs \tag{2}$$

where rhs is the constant produced by having assigned the first k variables already (and having moved this constant to the right-hand side). The next step in the solution procedure is to try all possible values for the next unassigned variable, x_k, and recurse with new values of k and rhs. The resulting algorithm in pseudocode is:

```
RecursiveSolver(used, k, rhs)
  if (k == n)   // if n vars assigned, check solution & stop recursion
    if (rhs == 0)
      PrintSolution(x,n)
    return

for each unused digit i (and avoiding i=0 if f_k=1)   // line A
  x[k] = i
  RecursiveSolver(used + 2^i, k+1, rhs - i*c[k]);
```

In this code, *used* is an integer that tells which digits (0-9) have been used in the variables assigned so far. *Used* is treated as an array of bits, with bit n being set if the digit n has been used, and can be employed to quickly determine the set of unused digits for line A.

This algorithm exhaustively tries all possible assignments of digits to variables. A significant optimization can be made by supposing that we can precalculate, for each k in $[0, n-1]$, the minimum and maximum value of the left side of (2) for any assignment of digits to the remaining x_i. Call these values min_k and max_k. If min_k is strictly larger than rhs for some value of k then we can terminate the recursion immediately, as no possible assignment of digits to the remaining variables can make the left-hand side of (2) small enough to be equal to rhs; a similar situation occurs if max_k is smaller than rhs. The following code, inserted before line A in the pseudocode above, implements this check:

```
if (min[k] > rhs) or (max[k] < rhs)
  return
```

This logic is an application of the "branch and bound" technique frequently used in integer programming. To increase its effectiveness it is useful to presort the variables so that $|c_0| \geq |c_1| \geq \ldots \geq |c_{n-1}|$, as this tends to maximize the size of the subtrees pruned from the search.

Here is the result of solving five alphametics with and without the min/max optimization.

Alphametic	***Exhaustive***	***Min/max test***	***Speedup***	***Time*** (μs)
SEND + MORE = MONEY	2085201	45	46338	1.3
ZEROES + ONES = BINARY	6905457	234	29511	5.5
COUPLE + COUPLE = QUARTET	7891281	131	60239	3.6
FISH + N + CHIPS = SUPPER	6182721	86	71892	2.3
BALLS + CLUBS + BALLET = JUGGLE	6904953	1935	3568	59.0

The first column of numbers gives the number of partial and complete variable assignments tried when using exhaustive search. The second column shows the number of assignments tried with the min/max check and third column is speedup factor (column1/column2). The table shows that this optimization is very effective, speeding up the search on average by a factor of about 42000 for these alphametics. The last column of numbers gives the actual runtime, in microseconds, of the algorithm (with the min/max test included, of course), implemented in C with a number of low-level optimizations for increased speed. Note that the execution times are almost linearly related to the second column of numbers, since the algorithm spends most of its time generating the trial assignments.

The min/max estimates are precalculated, before starting the recursion, using the method shown in the pseudocode below for max_k; the calculation for min_k is the same except that "if (c[i] > 0)" is changed to "if (c[i] < 0)".

```
MinDigit = 0
MaxDigit = 9
max[k] = 0
for i=k to n-1
```

```
if (c[i] > 0)
  max[i] = max[i] + MaxDigit*c[i]
  MaxDigit = MaxDigit - 1
else
  max[i] = max[i] + MinDigit*c[i]
  MinDigit = MinDigit + 1
```

These values are estimates of the true minima and maxima but, for example, min_k is guaranteed to be $\leq$ the real minimum m_k, so if $min_k > rhs$ then $m_k < rhs$, and thus the min/max tests work correctly. A tighter estimate of min_k and min_k could be computed at each step inside RecursiveSolver( ) but this requires more computation than is saved by the tighter bounds.

List Alphametics

An alphametic solver that can solve 100,000 or more alphametics per second can be used to find large examples of special types of alphametics. For example, the *list alphametic* is an alphametic of the form MEMBER1 + MEMBER2 + ... + MEMBERN = SETNAME, where the addends are members of some set of words and the sum is a word describing the set. Sets with a few hundred elements are small enough to be exhaustively examined to find the pure alphametic of that kind with the most addends. Three examples of provably-largest list alphametics are shown below, for the sets consisting of (a) the chemical elements, (b) Greek deities, and (c) world capitals. Can you find the unique solution for each of these?

```
     NEON        EOS      ACCRA   (Ghana)
   HELIUM       ARES      SANAA   (Yemen)
   OSMIUM       RHEA      PRAIA   (Cape Verde)
  LITHIUM       ERIS      PARIS   (France)
  HOLMIUM       HERA     MANILA   (Philippines)
 LUTETIUM       EROS     ASMARA   (Eritrea)
+SELENIUM      THEIA     MANAMA   (Bahrain)
 ELEMENTS      HADES     ASTANA   (Kazakhstan)
               MOROS    TALLINN   (Estonia)
              DEIMOS  +PRISTINA   (Kosovo)
              HESTIA   CAPITALS
              HERMES
              HEMERA
              AETHER
            +ARTEMIS
             DEITIES
```

Doubly-True Alphametics

A special kind of alphametic that is now commonly called a *doubly-true alphametic* was introduced in 1947 in a problem in *The American Mathematical Monthly* (Wayne, 1947) which asked for the solution to the alphametic FORTY + TEN + TEN = SIXTY. This alphametic is *doubly-true* because it is a valid sum when numbers are substituted for letters (the unique solution being 29786 + 850 + 850 = 31486) and also when its component words are read as numbers (40 + 10 + 10 = 60).

With a fast alphametic solver at our disposal it becomes possible to consider all possibilities up to some maximum value of the sum word (say, s) and thereby find all pure doubly-true alphametics up to some limit. To get an idea of the magnitude of this task, note that there are $p(s) - 1$ possibly-solvable doubly-true alphametics with sum word s, where $p(n)$ is the number of partitions of n. (The "minus 1" comes from the fact that there must be at least two addends, and so the singleton partition is eliminated.) Thus the total number of possible doubly-true alphametics with sum word $\leq s$ is

$$t(s) = \sum_{i=2}^{s} p(i)$$

For example, $t(100) = 1,642,992,467$ and $t(800) = 131,083,191,437,256,377,655,029,133,633$. However, the actual number of partitions that have to be generated and checked is much smaller than this, because for a given sum word s we only have to try partitions in which

a) all the addend words combined with the sum word use no more than ten letters of the alphabet (because we are working in base 10), and

b) no addend word has more letters than the sum word (because an alphametic with an addend longer than the sum cannot have a solution).

These observations reduce the number of possibilities by many orders of magnitude, and as a result we were able to exhaustively examine every possible partition for all sum words up to 800 in about 1000 CPU hours. Exactly 267,303 pure doubly-true alphametics were found, distributed as shown Table 1 below.

sum	*sum units*									
tens	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10		1								
20	13									
30	20									
40	94	67	57							
50	190	26	21		1	3	6	1		8
60	131	38	21			5	66			25
70	2089									
80	5336	56	61				2			11
90	36423	425	374	4	7	18	112	2	4	121
100	3									
140	7	6								
300	8									
340	7	6								
400	140									
440	7	7								
600	811									
700	278									
800	220184									

Table 1: The number of pure doubly-true alphametics for all sum words up to 800.

The sum s for a given cell in this table is given by adding the corresponding "sum tens" and "sum units" values. For example, the value 67 near the upper left corner is the number of pure doubly-true alphametics with sum $40 + 1 = 41$. If a cell is blank, or an entire row is missing (such as the sum tens = 110 row) then there are no alphametics with those sums. Note that a number > 100 (e.g. 261) is written as TWO HUNDRED SIXTY ONE.

Once we have these 267,303 puzzles stored in a file we can scan them to find examples with various interesting properties. For example, the puzzles with the fewest addends are these twelve (two with 3 addends and ten with 4 addends):

```
      SIX          TEN
    SEVEN          TEN
   +SEVEN       +FORTY
   TWENTY        SIXTY

    THREE         FIVE     SEVEN       SIX       SEVEN
    THREE         FIVE     SEVEN     EIGHT         TEN
    THREE         NINE     SEVEN     EIGHT         TEN
  +ELEVEN      +ELEVEN     +NINE    +EIGHT   +FOURTEEN
   TWENTY       THIRTY    THIRTY    THIRTY    FORTYONE

      TWO       ELEVEN       ONE      FIVE        FIVE
 FOURTEEN       ELEVEN      NINE      FIVE        FIVE
  FIFTEEN      FIFTEEN    TWENTY    THIRTY      THIRTY
  +TWENTY    +NINETEEN    +FIFTY    +FORTY      +FIFTY
 FIFTYONE     FIFTYSIX    EIGHTY    EIGHTY      NINETY
```

The tallest is this one, with 228 addends

```
THREEHUNDRED + ONEHUNDRED + NINETEEN + NINETEEN + EIGHTEEN +
EIGHTEEN + EIGHTEEN + THIRTEEN + THIRTEEN + TEN + NINE +
EIGHT + EIGHT + EIGHT + EIGHT + EIGHT + EIGHT + THREE + THREE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE +
ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE + ONE = EIGHTHUNDRED
```

An alphametic is said to be *ideal* if it uses ten different letters (and, therefore, all ten digits), so it is useful to categorize the 267,303 doubly-true alphametics by the number of different letters they have. There are 258,472 ideal puzzles with ten different letters, 8328 that use nine different letters, 444 using just eight, and only 58 that use seven, of which the simplest is TEN + TEN + TEN + TEN + NINE + ONE = FIFTY. And finally, there is exactly one pure doubly-true alphametic containing just five different letters:

```
TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO +
TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO + TWO = FIFTY
```

Note that the entries in Table 1 for the sums 140, 340, and 440 are all equal to 7. Why is this? A glance at the list of puzzles for these sums shows that they are all essentially the same, consisting of smaller puzzles to which the prefix ONEHUNDRED (or THREEHUNDRED or FOURHUNDRED) has been added to one of the addends and the sum, as in this example:

```
ONEHUNDREDTHREE
            TEN
           FOUR
           FOUR
          THREE
          THREE
          THREE
          THREE
          THREE
           +ONE
---------------
ONEHUNDREDFORTY
```

This works because the base puzzle (without the prefix) has only nine different letters and includes all the letters in ONEHUNDRED except for D. If the base puzzle is pure then so will be the extended one, with the single unused digit in the base puzzle being assigned to D. Also, the base puzzle has an addend with the same number of letters as the sum, so that the prefix can be added to both. The same set of seven base puzzles can also accept the prefix of THREEHUNDRED and FOURHUNDRED, which explains why the entries for 140, 340, and 440 are all 7. (Other xxxHUNDRED prefixes use too many different letters.)

The entries from 141, 341, and 441 are almost all the same, being 6, 6, and 7. The reason for the unexpected "7" is that one of the base puzzles is

```
FOURTEEN
   THREE
   THREE
   THREE
   THREE
   THREE
   THREE
   THREE
     ONE
     ONE
     ONE
     ONE
     ONE
    +ONE
--------
FORTYONE
```

which has some interesting properties. Firstly, all the letters except for F can be uniquely determined, and they use the digits 1 and 3-9, leaving 2 and 0 unused and F unassigned. Since F is a leading digit it cannot be 0, so it is 2 in the base puzzle. But when the prefix ONEHUNDRED is added, for example, then 0 is no longer disallowed for the value of F, so that F and D can be either 0,2 or 2,0. Thus the extended puzzle has two solutions and is no longer pure, so it does not appear in the list of pure puzzles for 141 or 341. But it *does* appear in the list for 441, because when the prefix FOURHUNDRED is added then F is once again a leading digit and cannot be zero, making the extended puzzle pure just like the base puzzle.

The technique of prefix extension raises an interesting question: what is the largest sum attainable in a pure doubly-true alphametic using extension? The best strategy seems to be to start with a base puzzle having nine different letters with one extra letter for the extension. The largest sum (nearly 10^{305}) that we have constructed in this way is shown below.

```
NINETYNINECENTILLIONNINETYNINENONILLIONNINETYNINEOCTILLIONTWENTYTWO
                                                          TWENTYTWO
                                                             TWELVE
                                                             TWELVE
                                                             TWELVE
                                                               NINE
                                                               +TWO
-------------------------------------------------------------------
NINETYNINECENTILLIONNINETYNINENONILLIONNINETYNINEOCTILLIONNINETYONE
```

Another interesting subtype of the doubly-true alphametic is one in which the addends are all different. These are very rare; of the 267,303 doubly-true alphametics up to sum=800 there are only twelve with distinct addends, with sums ranging from 51 to 91:

```
THREE + NINE + TEN + FOURTEEN + FIFTEEN = FIFTYONE
TWO + FOURTEEN + FIFTEEN + TWENTY = FIFTYONE
SIX + NINE + ELEVEN + SIXTEEN + NINETEEN = SIXTYONE
ONE + SEVEN + NINE + FIFTEEN + SIXTEEN + SEVENTEEN = SIXTYFIVE
FIVE + SIX + SIXTEEN + NINETEEN + TWENTY = SIXTYSIX
EIGHT + TEN + SIXTEEN + SEVENTEEN + EIGHTEEN = SIXTYNINE
FIVE + SEVEN + ELEVEN + TWELVE + FIFTEEN + TWENTY = SEVENTY
SIX + SEVEN + NINE + TWELVE + SIXTEEN + TWENTY = SEVENTY
THREE + SEVEN + TEN + TWENTY + THIRTY = SEVENTY
ONE + NINE + TWENTY + FIFTY = EIGHTY
ONE + TWO + FIVE + NINE + ELEVEN + TWELVE + FIFTY = NINETY
ONE + FIVE + TEN + ELEVEN + NINETEEN + FORTYFIVE = NINETYONE
```

The number of partitions of n into distinct parts is much smaller than $p(n)$, so a different program that searches just the space of distinct addends can be used to explore larger sums. We did not find any additional examples other than the twelve listed above (which were first constructed by Steven Kahan in the 1980's (Kahan, n.d.)). This suggests the following conjecture:

Conjecture. *The only pure doubly-true alphametics with distinct addends are the twelve shown above plus their prefix-extended versions.*

The pure doubly-true distinct-addend alphametic with the largest known sum (obtained by prefix-extension) is given in (Gonzáles-Morris, 2011) as

```
NINETYNINENONILLIONNINETYNINESEXTILLIONNINETEEN
                                        SIXTEEN
                                         ELEVEN
                                           NINE
                                           +SIX
-----------------------------------------------
NINETYNINENONILLIONNINETYNINESEXTILLIONSIXTYONE
```

Another type of doubly-true puzzle is the *two-sided doubly-true alphametic*, a puzzle like this one (which also has distinct addends):

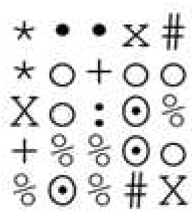
```
NINETY + NINE + ONE = FIFTY + FORTY + TEN
```

This puzzle is *balanced* because it has the same number of addends on the left and right side. Since a two-sided alphametic can also be formulated as a Diophantine equation we can use our fast solver to exhaustively search for large puzzles of this type. We discovered the currently largest known *balanced*, *pure*, *distinct-addend*, *two-sided doubly-true alphametic*:

```
(THIRTYNINE + THIRTYEIGHT + THIRTYTWO          (THIRTYTHREE + THIRTY + TWENTYNINE
 + THIRTYONE + TWENTYTHREE + TWENTY      =    TWENTYEIGHT + TWENTYTWO + TWENTYONE +
  + NINETEEN + THREE + TWO + ONE)                EIGHTEEN + TEN + NINE + EIGHT)
```

Other Varieties

Large examples of various types of alphametics can be found by generating many examples of the given type (either randomly or using a directed approach) and checking each one for pureness. Using this method a new record size (5x5) *spinning alphametic* was discovered:



A spinning alphametic has the following elegant property: it is a pure alphametic as written (using ten symbols rather than letters, with the top four lines being the addends and the bottom line the sum), and also (a) when rotated by 90°, (b) when rotated by 180°, and (c) when rotated by 270°.

A *word rectangle* is a rectangular arrangement of letters having a valid word in every row and column, such as these two (one of which also happens to be square), whose words are all found in the current Tournament Word List for Scrabble:

Each of these arrays, which we might call a *two-way alphametic word rectangle*, has a very special property: it is a pure alphametic as written and also when transposed. For instance,

```
SALAAM      BROOM
AXILLA      AORTA
LIMIER      BATTS
TSETSE      ACHES
            SHORE

 SALAAM       SALT
 AXILLA       AXIS
+LIMIER       LIME
 TSETSE       ALIT
             +ALES
              MARE
```

are both pure alphametics. We exhaustively checked all such alphametics that can be made using Scrabble words, finding 189 6x4 and 2395 5x5 arrays of this type, and no larger ones.

Another kind of extreme alphametic is the *alphametic* poem, an example of which is shown below. This poem, which describes and comments on the real-life saga connected with the novel *Jurgen* by Virginia author James Branch Cabell, is composed entirely of pure alphametics. Each line in the poem (and the title) is a separate alphametic, with the first $n-1$ words in each line being the addends and the last word being the sum. For example, the first line forms the pure alphametic

```
    HE
WRITES
    AT
+NIGHT
SERENE
```

Even one line of this poem could take many hours to compose by hand (the hardest part being to verify that the proposed alphametic is pure), but with our fast solver and some support software the effort can be reduced dramatically. For instance, when working on line 4, we might tentatively consider a phrase of the form

love and valor and X.

and then use a program which tries every possible word in some dictionary for X and returns those that result in a pure alphametic. In a fraction of a second this program prints the 359 words (all of them 6 letters) that can be used, from which the most pleasing one (we liked "chance") can be selected easily.

The Wiser Writer and the Inane Reader

He writes at night, serene.
No one, it seems, warms
 to his tales of ideal
 love and valor and chance.

As in times past he rearms
 again his genius: *Jurgen* ensues.

At first, no one sees; no one writes
 of his book or of his risks.
Then Mr. Sumner hisses
"Obscene!", and scene by scene bedecks
 Cabell's classic tale with witless
  remark and asinine fanfare.

There is a trial; fairly easily
is the case won. As the news echoes within
the land the tale sells and sell... and taunts
its author with odious
fervor for ever and ever. So fame, snarer
 of soul, takes its fitful
 victim again, as slings
  and arrows ever attend reason.

Enumerating All Non-Isomorphic Alphametics

If we consider all alphametics of a given size (a certain number of addends of specified lengths and a sum word of a given length) then an interesting question arises: how many distinct (in a sense to be described presently) alphametics are there?

Two alphametics should be considered equivalent under any permutation of the set of letters, because any such permutation will result in an alphametic with the same solution as the original. For instance, SEND + MORE = MONEY and DSYE + RMOS = RMYSN are isomorphic and both have the solution 9567 + 1085 = 10652. Two alphametics are also essentially the same if we permute the letters in one or more columns of the addends, because such a change will not affect the pureness of the alphametic, as it merely permutes the digits within the corresponding columns of the solution. So, for instance, MENE + SORD = MONEY is equivalent to SEND + MORE = MONEY since we have just swapped the letters in the left-most and right-most column of the addends; the solution 1565 + 9087 = 10652 has the digits in the same positions swapped.

To summarize: given an alphametic A, any alphametic B that is produced by permutation of the set of letters and/or permutation of one or more columns in A is isomorphic to A will be pure if and only if alphametic A is pure. To determine the pureness of all alphametics of a certain size it is sufficient to determine the pureness of the set of non-isomorphic ones.

The number of non-isomorphic alphametics of a given size can be determined by deBruijn's generalization of Pólya's counting theorem (see (Liu, 1968) and (Pólya & Read, 1987)), which counts the number of non-isomorphic ways of putting a single object (drawn from a set with c colors) into each of b boxes, with two configurations considered to be isomorphic under any permutation of the boxes from a group G and any permutation of the colors from another group H.

For instance, consider two-addend alphametics of this form:

```
          n
   ┌─────────────┐
     X Y Z W ... Q
   + A B X Y ... T
   ───────────────
   Y X Z W A ... W
```

We have $b = 3n + 1$ boxes and each gets a single ball having one of $c = 10$ colors. Since we consider two alphametics to be isomorphic under any permutation of the letters (colors), the group H is S_{10}, the symmetric group of order 10. For the group G, recall that we can swap the elements in any column of the addends (like X and A, Y and B, etc.). Thus the group G consists of the direct product of n copies of S_2 (from the n possible column swaps) and $n+1$ copies of S_1 (from the $n + 1$ letters in the sum, which must remain fixed); that is, $G = S_2^n \times S_1^{n+1}$. Applying deBruijn's Theorem with these G and H yields the following pretty theorem:

Theorem. *The number of alphametics with two n-letter addends and an $(n+1)$-letter sum is*

$$A(n) = \frac{1}{2^n \cdot 10!} \sum_{k=0}^{n} \binom{n}{k} (808640 + 410400 \cdot 3^k + 90720 \cdot 5^k + 25200 \cdot 7^k + 404460 \cdot 2^{m+k} + 136800 \cdot 3^{m+k} + 33600 \cdot 4^{m+k} + 6048 \cdot 5^{m+k} + 1260 \cdot 6^{m+k} + 240 \cdot 7^{m+k} + 10^{m+k} + 201600 \cdot 2^m 4^k + 56700 \cdot 2^m 6^k + 4725 \cdot 2^m 10^k + 60480 \cdot 3^m 5^k + 25200 \cdot 3^m 7^k + 18900 \cdot 4^m 6^k + 3150 \cdot 4^m 10^k + 5040 \cdot 5^m 7^k + 630 \cdot 6^m 10^k + 45 \cdot 8^m 10^k)$$

where $m = 3n - 2k + 1$.

The first few values of A(n) starting with $n = 1$ are 11, 371, 26683, 3375925, 667998741, 188471125067. The 11 non-isomorphic alphametics with $n = 1$ are these:

Pure	***Multiple Solutions***			***No Solution***						
A	A	A	A	A	A	A	A	A	A	A
+B	+B	+B	+A	+B	+B	+B	+A	+A	+A	+A
AC	CC	CD	BC	AA	AB	CA	AA	AB	BA	BB

We checked all the non-isomorphic alphametics of this form up to $n = 4$ for pureness; the results are shown in the table below.

n	**No. of distinct alphametics**	**No. of pure alphametics**	**No. w/multiple solutions**	**No. with no solution**
1	11	1	3	7
2	371	30	169	172
3	26683	1154	13696	11833
4	3375925	143726	1486810	1745389

A Huge Addition Alphametic

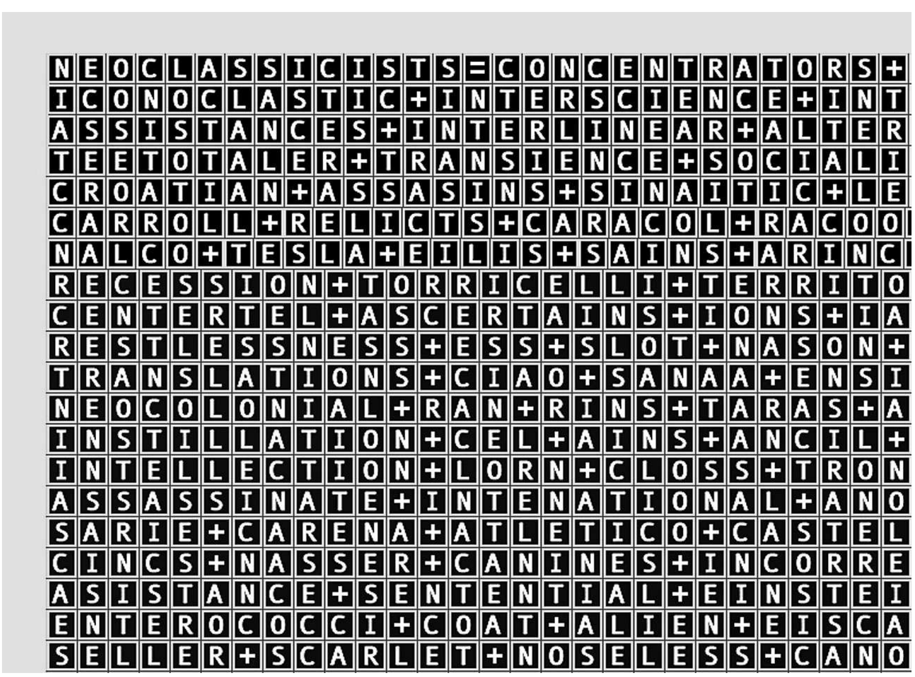
How many distinct English words can be arranged to make a pure addition alphametic? To tackle this challenge it is first necessary to answer a sub-question: for a given dictionary (we used Scrabble Tournament Word List), which set of 10 letters (with one of the letters disallowed as the initial letter in a word) permits us to form the largest number of words? Using a heuristic algorithm, but one that we think probably finds the optimal solution, we found that ACEILNORST, with O as the prohibited initial, spans the greatest number of words (11718).

To construct the actual puzzle, we use the following fact: when puzzles are very large they have the property that if there is a solution, it is almost certain to be unique. So we can actually assign digits to letters during puzzle construction, making it easy to guarantee that the puzzle has a solution, and then check it for uniqueness when it is fully built. Our algorithm for building the puzzle can be summarized as follows (we chose N=14):

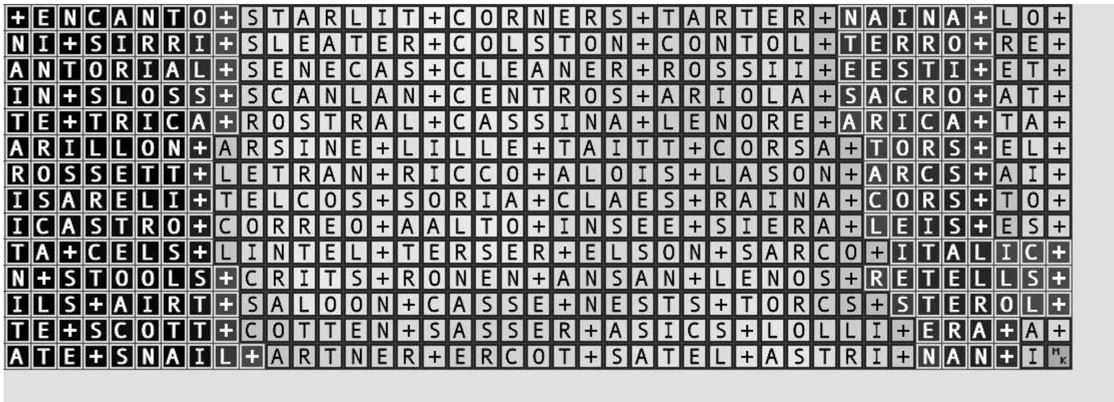
```
   sumword = a random N-letter word
   letter O = digit zero, assign the other letters to digits randomly
   sum = numerical value of sumword

   while not finished
[*]  pick an unused word, convert it to digits (call this 'value')
     sum = sum - value
     convert 'sum' from digits to letters (call this 's')
     if s is a valid, unused word, you're finished!
```

Basically, this algorithm repeatedly picks a new addend word and checks if the sum word minus all of the addends accidentally produces a number that when converted to letters is a valid, unused word. The tricky part is deciding how to pick each unused word in step [*]. If words chosen are too long on average then the value 'sum' will decrease too fast and the total number of addends used will be small. But if we choose words that are too small then when we get stuck and cannot choose any more words the partial sum will be large and therefore unlikely to accidentally be a word when converted to letters. Using a heuristic algorithm that works quite well, but could probably be improved further, we were able to construct a puzzle that uses 94% of all ACEILNORST words available.

Our best puzzle has 11024 addends ranging from 1 to 13 letters and the 14-letter sum NEOCLASSICISTS. Due to the size of the puzzle we thought it would be fun to try and display it in some interesting way. We decided to make each word, and its associated "+" or "=" sign, from a set of letter tiles (either all dark tiles or all light tiles for each word) and arrange the whole puzzle in a rectangle, with the words constrained to be in certain places so that the resulting array looks like a photograph from a distance. Here is a close-up of the upper left corner of the rectangle, which starts with the sum word "NEOCLASSICISTS =":



and here is the lower right corner:



The next page shows the alphametic viewed from a distance. The choice of image was directly inspired by the description of RM-II given in the colloquium announcement:

Our Colloquium will be a Show and Tell of bright pearls of Mathematics.

References

Dudeney, H. E. (July, 1924). *Strand* magazine, p. 97.

Wayne, Alan (1947). Problem E751, *American Mathematical Monthly*, Vol. 54, p. 412-414.

Kahan, Steven. Personal communication.

Gonzáles-Morris, G. (2011). In Donald Knuth, *The Art of Computer Programming Vol. 4A*, Addison-Wesley, p. 707.

Liu, C. (1968). *Introduction to Combinatorial Mathematics*, McGraw-Hill, p. 154-157.

Pólya, G., & Read, R. C. (1987). *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds*, Springer-Verlag, p. 109-112.

Words of 1-3 letters come from the current Scrabble Tournament Word List, words of 4 or more letters from Keith Vertanen's *wlist_match4* at `http://www.keithv.com/software/wlist/`.

Girl With An Alphametic Pearl

Mike Keith Feb 2011

Predominance Game

Helena Melo
Departamento de Matemática
Universidade dos Açores, Portugal
hmelo@uac.pt

João Cabral
Departamento de Matemática
Universidade dos Açores, Portugal
jcabral@uac.pt

Abstract

Predominance is a new strategy game for two-players, played on a Penrose's tiling board with a time control. The pieces are divided in white and black sets and each set has four different ranking pieces, ordered by value: commanders, knights, soldiers and conscripts. This game is played in two stages. The first stage is played by the two players putting in position the pieces on board, placed in an alternate way. In the second stage of the game, the players can move the pieces and make captures. The game ends when a player reach forty five points in captured pieces. Predominance is a cunning game, playable at several levels of complexity. Some winning strategies are explored.

Material

The game action happens on a Penrose's tiling board(fig.2).

The seventy two pieces are divided equally in white (fig.1) and black (fig.3) sets. Each set has four different ranking pieces, ordered by value: four commanders, six knights, ten soldiers and sixteen conscripts.

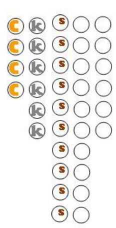


Figure 1: *White pieces*

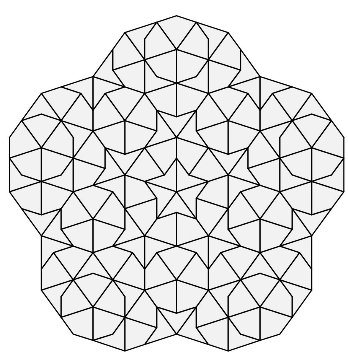


Figure 2: *Board*

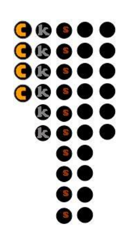


Figure 3: *Black pieces*

Rules

This game is played in two stages.

In the first stage the players put theirs pieces on board, placed in an alternate way. However, pieces of different colours can not occupy adjacent cells as pieces of the same category.

In the second stage each player moves their pieces.
They can make movements without capture or movements with intention of capture. The players are not obliged to capture.

The movement without capture (fig.4) is characterized by any path on the board, which is formed by only adjacent empty cells, with the maximum range of movement allowed for each piece.

The movement with the intention of capture (fig.5) is characterized by any path on the board, which is formed by any adjacent cells. So, this path can have non-empty cells. The capture is made replacing one piece, the captured, by the other, the captor, and the captor occupies the captured place.

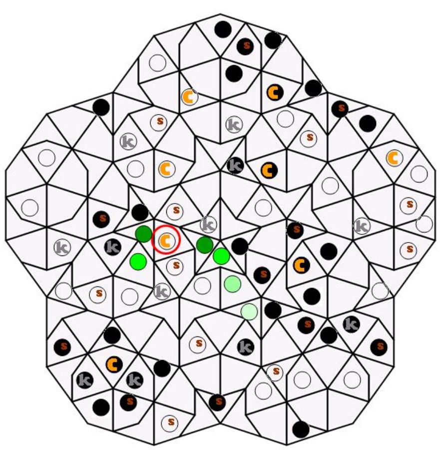


Figure 4: *Without capture*

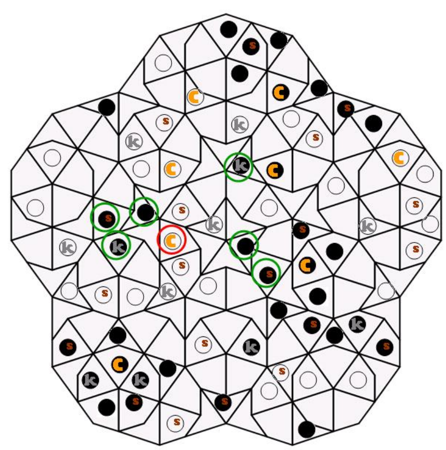


Figure 5: *With capture*

The piece can not move (fig.6) if it is blocked, i.e., it has not any possibility to capture or does not have any adjacent empty cell.

The commander moves the maximum of four adjacent cells. The knight moves the maximum of three adjacent cells. The soldier moves the maximum of two cells, and the conscript one cell.

The piece can only capture pieces of the same ranking or below.

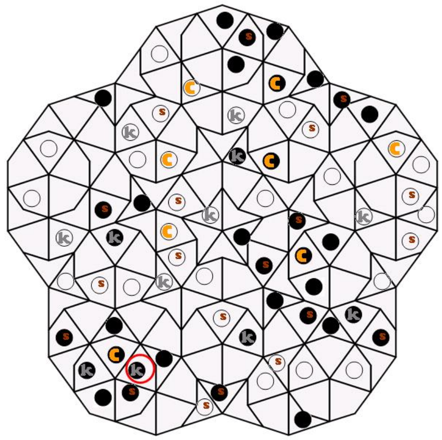


Figure 6: *Not move*

Goal

Each commander has the value of 8 points; the knight 4 points; the soldier 2 points; the conscript 1 point.

The game ends when a player reaches forty five points in captured pieces.And the first player achieving this value is declared the winner.

For example, if one player captures two commanders, three knights, five soldiers and seven conscripts, then he has forty five points.

Playing the game

In this section we present an example of the action. To simplify, we will only demonstrate a few movements.

The game starts with the white player.

At movement 1 (fig.7) the white knight captures the black knight, and the player obtains 4 points. Then we have a new position (fig.8) of the pieces on the board.

Now is the black player's turn. At movement 2 (fig.9) the black commander captures the white commander, and the black player obtains 8 points.Then we have a new position (fig.10) of the pieces on the board.

Now is the white player's turn. At movement 3 (fig.11) the white commander captures the black commander, and player obtains 8 points. Then we have a new position (fig.12) of the pieces on the board. He has 12 points in this moment. And so on... until one of this players complete 45 points.

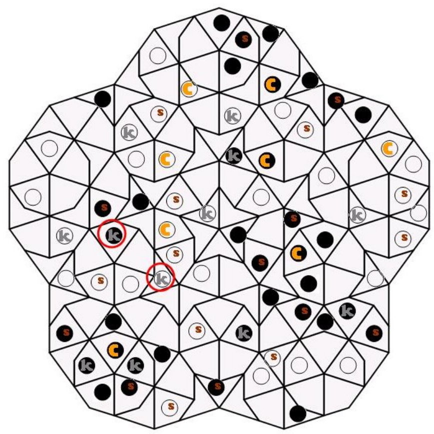


Figure 7: *Movement 1*

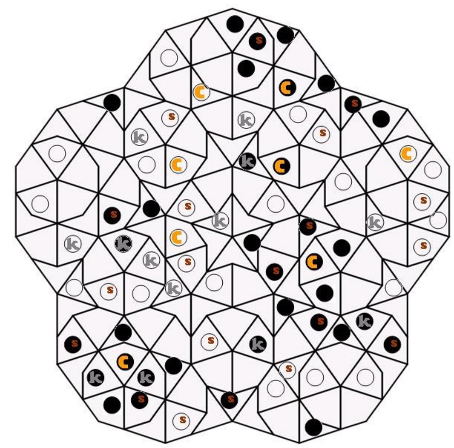


Figure 8: *New position*

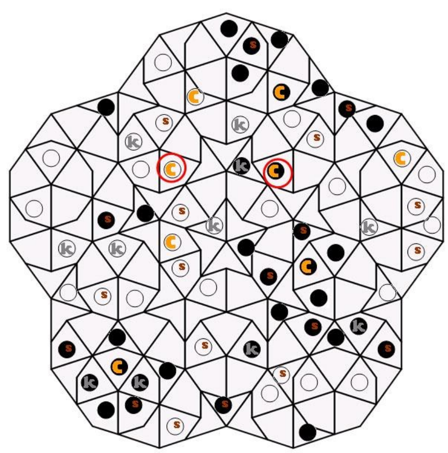


Figure 9: *Movement 2*

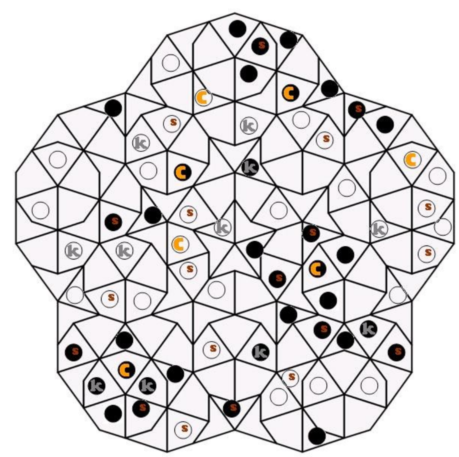


Figure 10: *New position*

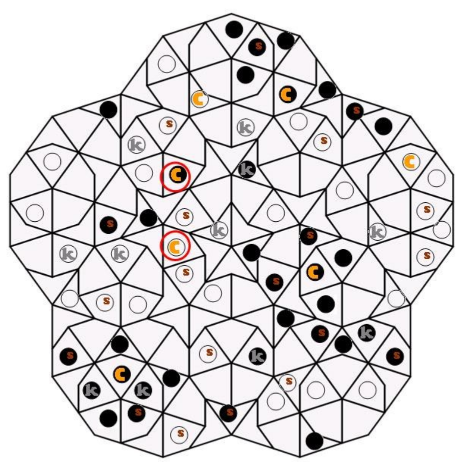


Figure 11: *Movement 3*

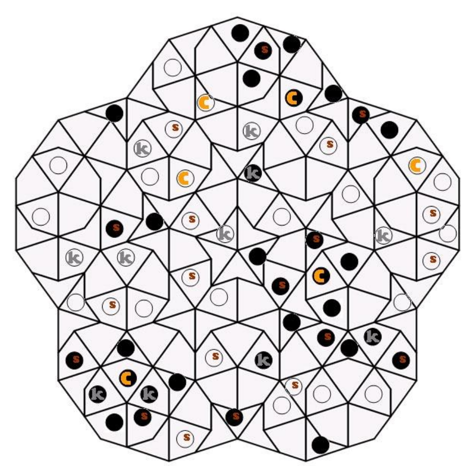


Figure 12: *New position*

The mathematics

To initiate the study, we will research the best position for each piece.
But, first of all, we need to understand better the board. We have six zones demarcated as showed in figure 13. Each zone has 5, 10, 20, 20, 40 and 50 cells (fig.14), respectively, starting from the interior to the border. The board has the total of 145 cells.

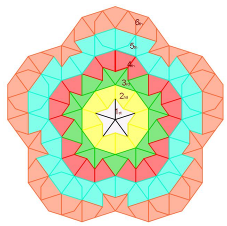


Figure 13: *Board*

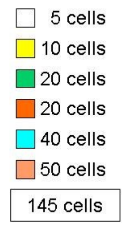


Figure 14: *Cells*

The board is composed of five equal parts by rotation, and it has a symmetry axis. So, we can put the cells in sequence considering this two symmetries. The figures 15, 16 and 17 show this aspect. This sequence, beginning from the center, can be observed in figure 18. We start our study using it.

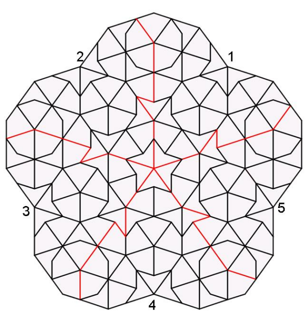


Figure 15: *Rotation*

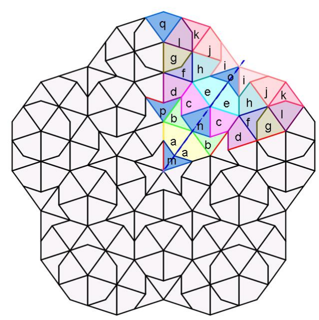


Figure 16: *Symmetry Axial*

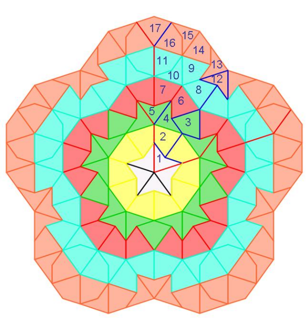


Figure 17: *Sequence*

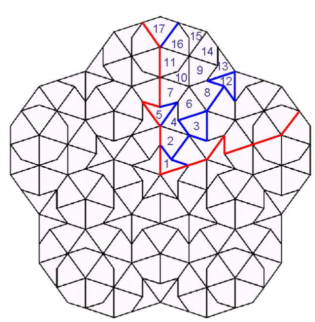


Figure 18: *Board with order*

We obtain the following results.

Let's put a piece on the black cell in figure 19. If it is a conscript the zone of action is the colour pink. If it is a soldier the zone of action is the colour pink or blue. If it is a knight the zone is the colour pink, or blue or orange. If it is the commander, the zone of action is all colours. Under the figure we have a table with the correspond numbers of cells in each action. The other table, under the figure, shows the possibilities of dislocation for each piece rank.

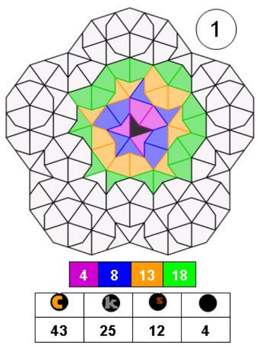


Figure 19: *Position 1*

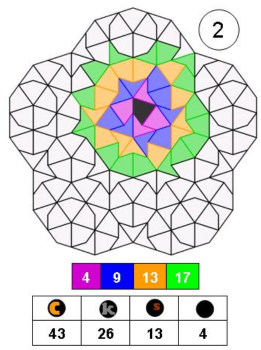


Figure 20: *Position 2*

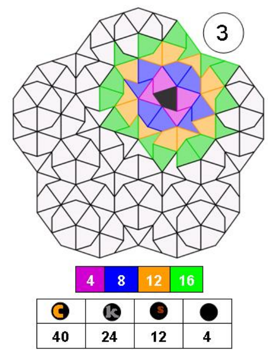


Figure 21: *Position 3*

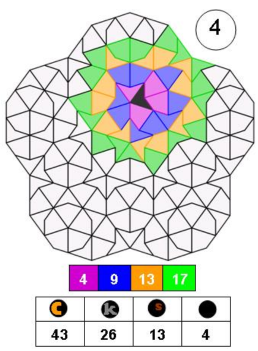


Figure 22: *Position 4*

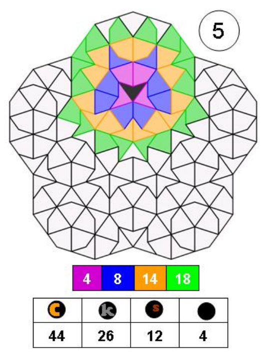


Figure 23: *Position 5*

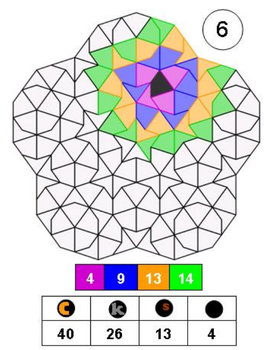


Figure 24: *Position 6*

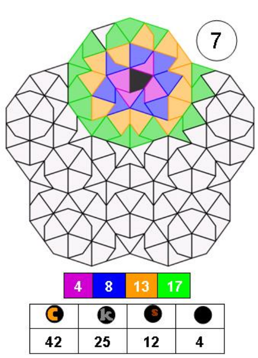


Figure 25: *Position 7*

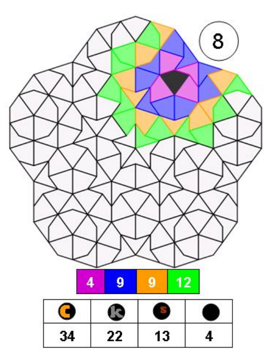


Figure 26: *Position 8*

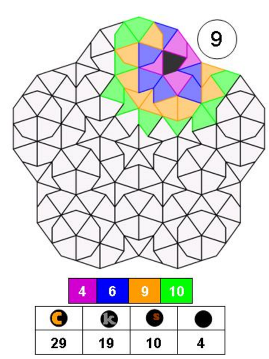


Figure 27: *Position 9*

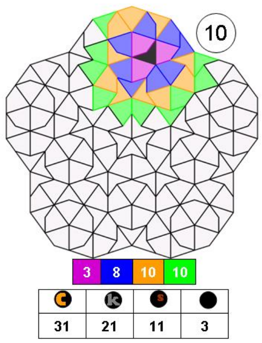


Figure 28: *Position 10*

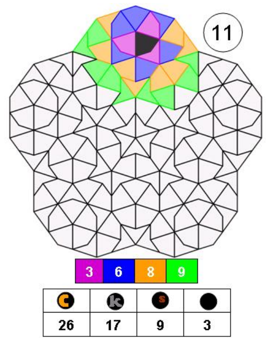


Figure 29: *Position 11*

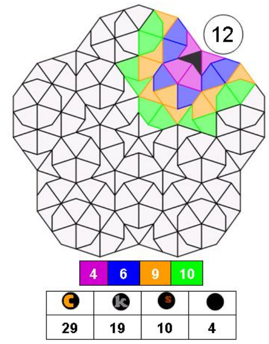


Figure 30: *Position 12*

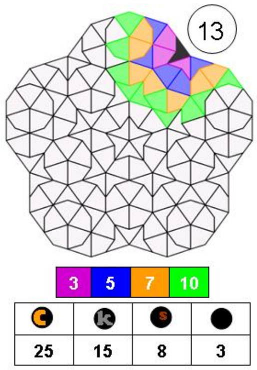


Figure 31: *Position 13*

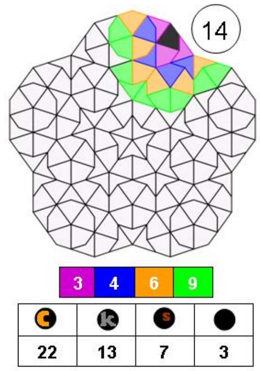


Figure 32: *Position 14*

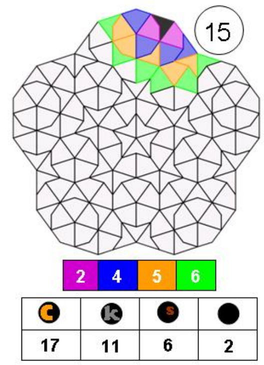


Figure 33: *Position 15*

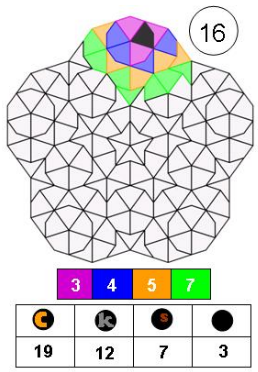


Figure 34: *Position 16*

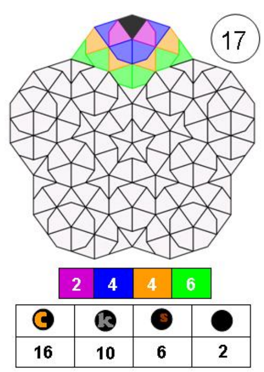


Figure 35: *Position 17*

Analyzing all our possible positions, and resuming them in a table 1, we can observe the best position (table 2). The results resumed in table 2 allocated on the board, we have the best position for the commander (fig.36), the knight (fig.37), the soldier (fig.38) and the conscript (fig.39). Of course, this is an ideal position not considering the position of the others pieces.

*	C	K	S	c
1	43	25	12	4
2	43	26	13	4
3	40	24	12	4
4	43	26	13	4
5	44	26	12	4
6	40	26	13	4
7	42	25	12	4
8	34	22	13	4
9	29	19	10	4
10	31	21	11	3
11	26	17	9	3
12	29	19	10	4
13	25	15	8	3
14	22	13	7	3
15	17	11	6	2
16	19	12	7	3
17	16	10	6	2

Table 1: *All possibilities*

C	K	S	c
5	2,4,5,6	2,4,6,8	1-9,12
1,2,4	1,7	1,3,5,7	others
7	3	10	15,17
3,6	8	9,12	-
8	10	11	-

Table 2: *The best position*

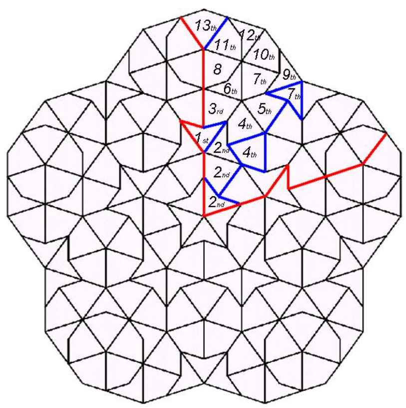


Figure 36: *Best position for the commander*

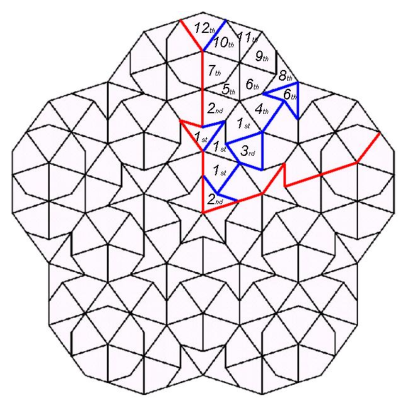


Figure 37: *Best position for the knight*

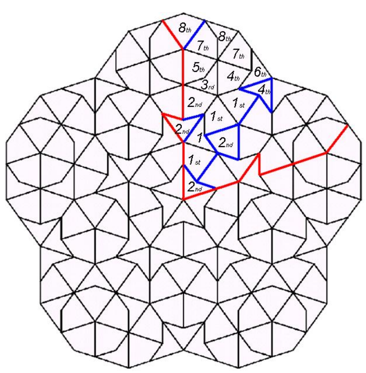


Figure 38: *Best position for the soldier*

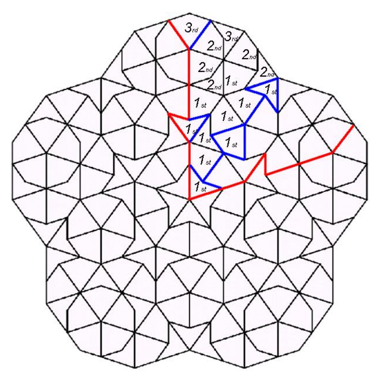


Figure 39: *Best position for the conscript*

Bocage and Mathematics

Filipe Lopes Papança
Military Academy - Portugal
filipe.papanca@gmail.com

Abstract

This communication analyzes the Mathematic knowledge in some poetry of Bocage and the politic and scientific context in the second part of the eighteenth century. Also, it analyzes the influence of his military formation in the acquisition of Mathematic Knowledge.

Keywords: Bocage, Mathematic Knowledge, military formation.

Setúbal - House who was born Bocage painting by Alberto de Sousa

This communication was inspired by a invitation made by the poet *Maria de Deus Melo* to write a literary reflection to include in this book *Eternamente Bocage* edited by *Universitaria Editora*. This reflexion utilised some references of my doctoral thesis *Mathematics, statistics and teaching in establishments who train Army Officials the period 1837-1926: a characterization* defended in Évora University in 7 June of 2010 and edited by EdiumEditores.

Bocage (1765-1805) was one of most popular Portuguese poets. This influence in the Romantic Movement was very important. When I was a child I often listened to my grandmother Evangelina Lopes say his poetry.

He had a military formation - *Marine Academy* and *Marine Guard Academy* establishments who formed military and marine officials when he received Mathematical formation. In the Marine Academy in this time they taught arithmetic, calculus, trigonometry and its applications in several domains like optical. He made also humanistic studies in the *Royal Class* of the Spanish priest D. João de Medina. His grandmother was a great writer and translator.

When we analyse his poetry we can find Mathematical, geometrical and astronomic references in some poems like *Satire to Filinto*, poet of Arcadia literary movement the true name Francisco Manuel do Nascimento (1734-1819), priest, translator, very popular in the army (Artillery Arm) when in metaphorical sense he writes *Nose, what Newton don't want describe his diagonal.* The mathematical (calculus), physical (mass) and astronomical knowledge (eclipses) was utilised to describe in hyperbolic form, the nose.

It was a time when astronomy with the introduction of spherical trigonometry was a great development. They discovered unknown stars in base of declination calculus, described the comets orbits and foresaw the eclipses. We can see the description of these methods in the books of the Jesuit priest Monteiro da Rocha, Latin Grammatical and rhetoric teacher who becomes an academic and chancellor at Coimbra University with extinction of Jesuit congregation by the pope influenced by Pombal Marquis, Secretary of State of King Joseph first of Portugal. In Évora in the libraries years ago was rediscovered the book *Mathematical Fisical system of Comets* of Monteiro da Rocha.

In Mathematics the recent discovery of Infinitesimal Calculus by Newton and Leibnitz had a decisive role. In the library of Military Academy - Lisbon we can find the book *Aritmética Universal* of Newton (1802). We can conjecture its utilization in military teaching.

Also the philosophy flourishes with Voltaire, Rousseau, the encyclopaedism inspired by d' Alembert and chemistry with Lavoisier experiences.

In the political plan we feel the ideal of the French Revolution. In his poetry he utilizes the antithesis freedom - despotism well expressed in poem named *Marília letter*. Because of that he stayed some time in prison (Limoeiro and Inquisition custody).

By the same reason other poets of this time have the same destiny like Anastácio da Cunha (1744-1787) also Mathematician, Artillery Official, Academic (Coimbra University) by the power of Pombal Marquis and after the prison became director of Pia House (institution that protects abandoned children and because of that directs several colleges like S.Lucas in S. Jorge's Castel -Lisbon) by Pina Manic, Police Superintendant in the time of Queen Mary first of Portugal. In the Regimental School of Valença, reorganized by the Prussian Artillery Official Schaumbourg Lippe he learned Mathematics and meets some foreign officials aimed at the new political ideas and he can read books not mentioned in the school curriculum (because of that he received a reprimand). The Mathematical books of this author mentioned in references are also in the library of Military Academy - Lisbon. We can conjecture the influence of Anastácio da Cunha, Masonic ideas and the astronomic knowledge of Monteiro da Rocha in Bocage.

In Bocage poetry it also refers to Goa where he was because of his military career. But he didn't like it. In a poem about this city he says there was more vanity than in London, Paris, Lisbon.

As a Bohemian, he prefers the literary meetings, parties, games with money and other distractions to the studies. He abandons the military life and becomes a vagabond. He becomes a legend because his poetry allied to his behaviour made scandal.

In this poetry there was a permanent conflict between reason and love. His numerous passions are a reflex of a soul that prefers the muse cult to the natural sciences. Finally he found some conciliation. He felt consolation in this sentence of S.Matheus *my yoke is easy my burden is light.*

He died in 21 December 1805. In the same day of 1872 was inaugurated in this town Setúbal the monument in his memory. In the last days with the perfect knowledge of French and Latin languages he gave life to many other poets and writers.

References

Antunes, J. R. (1886). *Apontamentos para a historia da Escola do Exército*. Lisboa: Imprensa Nacional.

Barata, M. T., & Teixeira, N. S. (Eds.). (2004). *Nova história militar de Portugal*. Lisboa: Círculo de Leitores.

Boyer, C. (1999). *História da Matemática* (2ª Edição). São Paulo: Blucher.

Carvalho, R. (2001). *História do ensino em Portugal* (3ª Edição). Lisboa: Fundação Calouste Gulbenkian.

Cunha, J. A. (1790). *Principios Mathematicos para instrucção dos alumnos do Collegio de S. Lucas da Real Casa Pia do Castello de S. Jorge*. Lisboa: Officina de Antonio Rodrigues Galhardo.

Cunha, J. A. (1987). *Princípios Matemáticos para Instrução dos Alumnos do Collegio de S.Lucas da Real Casa Pia do Castelo de S. Jorge*. Coimbra: Departamento de Matemática da Universidade. Reprodução fac-simile da edição publicada em Bordéus em 1811.

Gonçalves, V. (1940): Análise do Livro VIIII dos Princípios Matemáticos de José Anastácio da Cunha. Em *Discursos e Comunicações apresentadas ao Congresso da História da Actividade Científica Portuguesa* (VIII Congresso), 12º Vol., Tomo 1º, 1ª secção. Lisboa: Congresso do Mundo Português Publicações.

Melo, Maria (2006). *Eternamente Bocage*. Lisboa: Universitária Editora.

Newton, I. (1802). *Aritmética Universal*. Paris: Bernard.

Rocha, M. (1808). *Memoires d'Astronomie Pratique*. Paris: Coucier.

Rocha, M (2000). *Sistema Físico-Matemático dos Cometas*. Rio de Janeiro: Museu de Astronomia e Ciências Afins.

Sampaio, R. A. (1991). *História da Academia Militar*. Lisboa: Academia Militar.

Sena, C. (1922). *A Escola Militar de Lisboa: História, Organização, Ensino*. Lisboa: Imprensa Nacional de Lisboa.

Simões, J. M. O. (1892). *A Escola do Exército*, Breve Noticia da sua História e da sua Situação Actual. Lisboa: Imprensa Nacional de Lisboa.

Saraiva, A e Lopes, L. (Eds.). (1979). *História da Literatura Portuguesa* (11ª Edição). Porto: Porto Editora.

Selvagem, C. (1931). Portugal Militar. Lisboa: Imprensa Nacional de Lisboa.

Struik, D. J. (1999). *História concisa das matemáticas*. Lisboa: Gradiva.

Math Games

Maria de Fátima Rodrigues
Centro de Estruturas Lineares e Combinatórias, UL
Department of Mathematics, FCT/UNL
ClubeMath, FCT/UNL - clubemath@fct.unl.pt

Maria do Céu Soares
Centro de Matemática e Aplicações, UNL
Department of Mathematics, FCT/UNL
ClubeMath, FCT/UNL - clubemath@fct.unl.pt

Nelson Chibeles-Martins
Centro de Matemática e Aplicações, UNL
Unidade de Modelação e Optimização de Sistemas Energéticos do LNEG
Department of Mathematics, FCT/UNL
ClubeMath, FCT/UNL - clubemath@fct.unl.pt

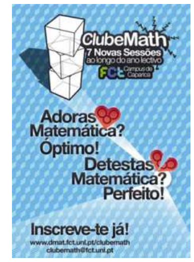
"I always believed that the best way to make mathematics attractive to students and people in general is showing it like a game ..."

Abstract

In this work we will introduce *ClubeMath*, followed by a brief explanation on why we use games as a tool and simultaneously as a vehicle for awakening and raising mathematical awareness. Thirdly, we will present several games developed during our regular activities and other games created under the "Recreational Projects in Mathematics" workshop aegis.

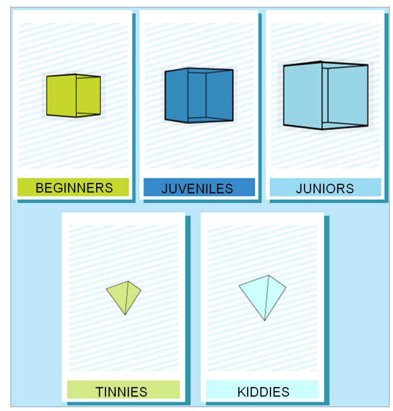
ClubeMath

According to António Aniceto Monteiro (Monteiro, 1942) "Math Clubs aim to promote and develop a taste for the study of mathematics among high school and university students, emphasizing, in meetings especially designed for this objective, the beauty of this science and the usefulness for modern life of its learning. In addition, Math Clubs are powerful teaching aids, and support the cultural and moral formation of its members." Following Monteiro's view, *ClubeMath* – the Club of Mathematics of FCT/UNL (Faculdade de Ciências e Tecnologia/ Universidade Nova de Lisboa) is, essentially, a Club that aims to show a different side of Mathematics, through fun and recreational activities, in order to stimulate skills and interest in this science. It was born in June 2007, but its main objectives - to fight the social phenomena of "I hate math!", and to join in the same room young people with the common interest "I love math!" – remain updated.



At *ClubeMath* we develop recreational mathematical activities in order to deny the widespread notion that mathematics has an inherently complex and inaccessible nature. Simultaneously, we think this kind of activities have the ability to generate a vital stream of interest in mathematics and, therefore, to increase gradually the empathy with this area.

ClubeMath has three different fields of action. Firstly, the activities for associates (which are students of basic and high school levels), for whom we develop seven afternoon sections in each school year, and the ones for their family, as their key influencers. Secondly, the actions directed towards schools, either through receiving their students in FCT/UNL or by forming their teachers in the context of their continuum formation. Last, but not least, the outreach activities for the general community, such as the organization of conferences, or the development of a marketable math board game.



***ClubeMath* Games**

Why Games?

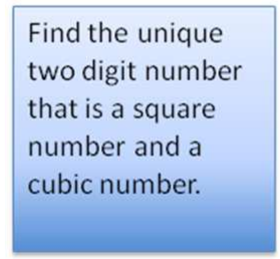
"Playing games is a type of activity that combines reasoning, strategy and reflection with challenge and competition in an extremely rich and entertaining fashion. Team games may also favour co-operation. Playing such games, in particular strategy, observation and memory games, contributes towards the development of mathematical skills as well as personal and social development." (Portugal's National Curriculum of Basic Education - Essential Competences).

In *ClubeMath* we try to apply the aforementioned principles directly, organizing games with mathematical, logical, strategic and social components. Our games are usually played by teams of players in order to stimulate interpersonal cooperation. Moreover, games are a funny and easy way to capture the target audience's attention, when subjects usually considered arid are introduced.

In the following we will present some *ClubeMath* games.

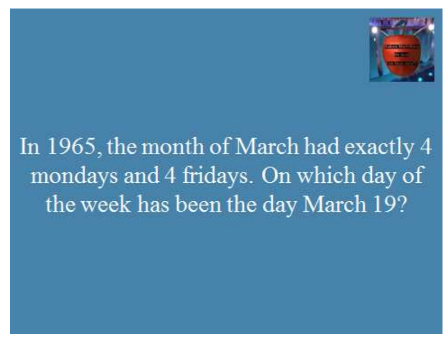
BingoMath

BingoMath plays like a regular Bingo game, but one in which the host, instead of announcing a number, presents one of a list of ninety numerical problems. The solution to each problem is an integer between 1 and 90, to be marked on the player cards. This game, originally developed with a wide variety of math questions, can be easily adjusted to almost any particular item of math's curriculum as, for instance, resolution of equations, trigonometry or even geometry. This versatility is present in several others *ClubeMath*'s games.



Sabes MathMais do que os teus Pais? (Do you know MathMore than your Parents?)

This is a Parents & Kids game, in which teams of players (the associates) can be aided or saved by a group of Parents. The host proposes a question and the players receive 5 points if they can answer without any Parent's aid and 0 points if the answer is wrong. Any playing team can apply for a particular aid once in a game, using at most one aid in each question. The available aids are:



- Advise – A Parent chosen by the playing team suggests an answer that the team can ignore. If the team answers correctly, it receives 3 points, otherwise, looses 3 points;
- Impose – A Parent chosen by the player team answers the question and the team must use the answer. The correct answer awards 3 points, but a wrong one does not penalize the team;
- Opine – All Parents advise the team, who can ignore all the suggestions. If the team answers correctly, it receives 3 points, otherwise, looses 3 points;
- Rescue – The team gives a wrong answer. All Parents advise the team, who has to choose one of the suggestions. If the team answers correctly it receives 3 points, otherwise, looses 5 points;

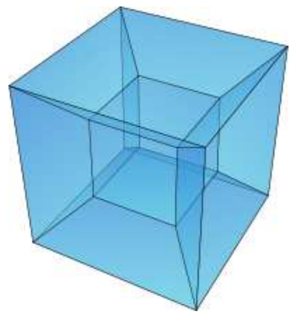
Besides the obvious benefits concerning math knowledge, these games also improve kids & parents cooperation, in a very fruitful intergenerational challenge.

Assalto às MathMasmorras (Dungeons & MathDragons)

This game is loosely inspired by the famous RPG Dungeons & Dragons ©. It is a Cooperative game where each participant incarnates a hero with strong and weak characteristics. The team faces deadly traps, creepy monsters and cryptic puzzles, trying to discover the secret hidden in the center of the dungeon. The mathematical traps and puzzles are intended to introduce new concepts and/or recall concepts already learned, such as geometry, graph theory, combinatorial calculus and cryptography.

For instance, during a dragon hunt the players discover the following message written using the Caesar's Code: ***B qentñb r' qr pbe oenapn. Abeznyzragr fñb ihyaren'irvf nb sbtb r pbfcrz tryb...*** The team must research what the Caesar's Code is and try to de-codify the message: ***O dragão é de cor branca. Normalmente são vulneráveis ao fogo e cospem gelo...*** ("The dragon is white. Usually, they're vulnerable to fire and breathe ice")

In other level, the exploration of a maze can be the occasion to introduce the concept of the tesseract.



Tetr4Math

Board game containing questions in four different fields:

- *General Topics*, located in the white squares and associated with the letter **T**,
- *Numeric Expressions*, to be found in the light blue squares, and associated with the letter **E**,
- *Geometric Theory*, located in the green squares, and linked to the letter **T**,
- *Reasoning and Logic*, within the dark blue squares and related to the letter **R**.

tetr4math

To win the game, the player must also collect the letter **A**, that completes the word **TETRA**.

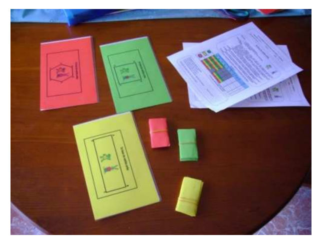
This is also a game of strategy, because any player can decide, in his turn, either to proceed forward or to rotate the board in order to change the relative placement of all players and their chances of getting all 4 letters.

Games in school use

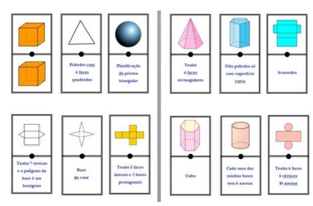
It is currently in progress an MSc thesis, where it will be compared the students' reaction to the same question in two different contexts: in the framework of a *ClubeMath* game, and within an evaluation test (R. Botelho, FCT/UNL). This comparative study, that we expect will be finished in September 2011, will be done on two different programmatic contents of the 7^{th} (Portuguese) grade, namely statistics and the resolution of first degree equations.

Although there are still no conclusions from the above study, we believe that games can be and, in fact, should be applied in the context of a math classroom, both for simply recreational purposes or, namely in the context of reviewing mathematical knowledge, as an attempt to counteract the lack of interest and motivation that usually surrounds this subject. With this objective in view, we developed the workshop "Recreational Projects in Mathematics" (Portuguese Scientific and Educational Training Office accreditation CCPFC/ACC - 57129/09), directed at basic and high school teachers. This formation action aimed to give those teachers the necessary resources in order to be able to develop their own creative projects and the related ludic materials. Several extremely interesting games were created by the trainees of this workshop, and applied in their math classrooms. We briefly describe three of them, designed for 5^{th}-grade math classes:

– *Searching for the right measures*
This game consists on searching for the "right measures" of several polygons (square, rectangle and hexagon), exploring the relationship between the length of their sides and its perimeter. In each question card, there were three different levels of increasing difficulty: green, yellow and red.



– *Geometric domino*
With this domino the students were able to apply all their knowledge on elements of geometric solids (faces, edges, vertices), classification of geometric solids, elements of a polygon (sides, vertices), classification of polygons, and planified solids.



– *Slow and steady wins the race*
Besides the geometric component already present in the two previous games (distinguishing between polyhedron and non polyhedron, identifying elements of geometric solids or identifying polygons) this board game also had a second kind of questions, devised to foster mental calculation, with integer and decimal numbers.

At the end of the math class, both students and teachers reported their opinion on what had been achieved through this activity. The results strongly indicate that this kind of activities are not only motivating for further learning, but also facilitate the retention and systematization of contents.

Conclusions

In this work we presented *ClubeMath*, its objectives and main activities. Several different mathematical games developed during *ClubeMath*'s regular sessions and other parallel actions were introduced, and their purposes and characteristics were presented. We are guided by the strong belief that games can be thoroughly used as a tool for increasing the awareness of Mathematics' importance, in particular among children and adolescents. Games are funny, can induce cooperative work, increase concentration and defuse the idea that learning math is a chore. Notwithstanding this conviction, we do not advocate the replacement of formal learning with ludic approaches. On the contrary, we defend math teaching should emphasize rigor and formality. But games can and should be used, outside the framing of traditional school activities, as a way to bridge the span between the young with a negative disposition towards mathematics and this important, very interesting, although demanding, subject.

References

Chibeles-Martins, N., Rodrigues, M.F., & Soares, M.C. (2010). *Adoras Matemática? Óptimo! Detestas Matemática? Perfeito!*, Boletim da Sociedade Portuguesa de Matemática, Actas do Encontro Nacional da SPM 2010, pp 124-129.

Monteiro, A. A. (1942). *Clubes de Matemática*. Gazeta da Matemática (11), pp. 8-12.

Portugal's National Curriculum of Basic Education – Essential Competences: `http://sitio.dgidc.min-edu.pt/recursos/Lists/Repositrio%20Recursos2/Attachments/95/national_curriculum.pdf`, pp 69.

Bodies Invisible in Several Directions

Alexander Plakhov
Department of Mathematics
University of Aveiro
Aveiro 3810-193
Portugal

Vera Roshchina *
CIMA
University of Évora
Portugal

Abstract

We give an overview of the recent findings in the invisibility in billiards and discuss open problems in this area.

Introduction

Invisibility is an inherently fascinating concept whose place, until recently, has been confined to fairy-tales and myths. The place of the invisibility cloak has been alongside other highly desirable but non-existent items, such as magic mirrors, flying carpets and seven-league boots, but this picture is changing rapidly now.

Imperfect, but still impressive versions of an invisibility cloak can be produced nowadays with the use of cameras to project the image from behind on a specially designed surface (e.g. see Inami, Kawakami & Tachi, 2003). More fundamental approaches focus on constructing materials with special refractive properties. Such metamaterials do not exist in nature, however, they can be engineered. For example, following on from the work (Schurig, Mock, Justice, Cummer, Pendry, Starr & Smith, 2006), researchers at Duke University have demonstrated a body invisible in microwaves (see Chang, 2007); in (Valentine, Zhang, Zentgraf, Ulin-Avila, Genov, Bartal & Zhang, 2008) construction of a 3D optical metamaterial with a negative refraction index was reported; in (Ergin, Stenger, Brenner, Pendry & Wegener, 2010) metamaterials are used to hide a bump in a metallic mirror.

The effects specific to geometrical optics, however, also remain important in modern technology, mostly in cases where the objects are large enough for geometrical optics to dominate the wave effects, such as, for example, the design of lenses (e.g. for photography or DVD readers) and fiber optics.

This paper gives an overview of recent findings on the invisibility of billiards. In particular, we discuss the construction of bodies invisible in one, two and three different directions and discuss

* Ciência 2008

some open problems. Our exposition is primarily based on the works (Aleksenko & Plakhov, 2009), (Plakhov & Roshchina, 2011a) and (Plakhov & Roshchina, 2011b): we refer the reader to them for technical details and proofs.

Invisibility in a direction v means that any light ray which initially moves along a straight line in this direction, after several reflections from the body's surface will eventually leave the body and keep moving along the same straight line. Invisibility in a set of directions means that the above is true for any direction from this set. This problem is closely related to the problem of minimal resistance going back to Newton (Newton, 1687). The latter consists of finding a body, from a given class of bodies, that experiences the smallest possible force of flow pressure, or resistance force. Since the 1990s, many interesting results on this problem have been obtained by various authors (see, e.g., Bucur & Buttazzo, 2005; Buttazzo & Kawohl, 1993; Comte & Lachand-Robert, 2001; Lachand-Robert & Oudet, 2006; Lachand-Robert & Peletier, 2001; Plakhov, 2009; Plakhov & Aleksenko, 2010).

The rest of the paper is organized as follows: in Section *Polyhedral bodies invisible in one direction* we talk about polyhedral constructions invisible in one direction, then, in Section *Parabolic constructions and invisibility in two directions* introduce parabolic bodies that ultimately lead to the design of bodies invisible in two directions, and, finally, in Sections *A thin fractal body invisible in two orthogonal directions* and *A body invisible in 3 directions* talk about fractal bodies that allow invisibility in two directions in two-dimensional settings and three directions in three-dimensional settings respectively. Section *Summary* contains a brief discussion of the open problems.

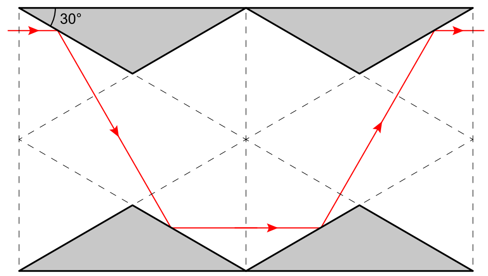
Polyhedral bodies invisible in one direction

As we have mentioned in the introduction, invisibility in a direction v means that any particle which initially moves along a straight line in this direction, after several specular reflections from the body's surface will leave the body and keep moving along the same straight line.

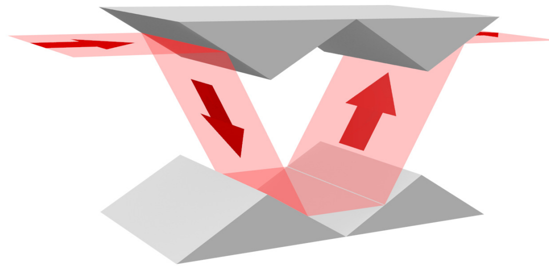
In general, it is possible that the particle never leaves the body and keeps bouncing off its sides infinitely; however, we only consider such bodies and initial directions for which almost every particle makes a finite number of reflections. Also note that in some cases the particle may hit a singular point of the boundary. In this case the further movement of the particle is not defined. We consider such bodies and initial velocities for which the set of points with undefined motion has zero measure.

Observe that invisibility is a symmetric notion, i.e. if a body is invisible in a direction v, it is also invisible in the opposite direction $-v$. This follows directly from the fact that billiard dynamics is time-reversible.

The body shown on Fig. 1 (a) is constructed using four equilateral triangles (dashed lines). The two-dimensional body that is shown in grey is invisible in the horizontal direction. This can be worked out using elementary geometry (see (Aleksenko & Plakhov, 2009)). Indeed, consider a particle moving horizontally from left to right. After the first reflection (either from the top or the bottom of the construction) the particle is redirected at $60°$ from the vertical direction, it symmetrically hits the opposite side, and after that keeps moving in the original rightward horizontal direction. The symmetric construction on the right then redirects the particle back to the original trajectory.



(a)



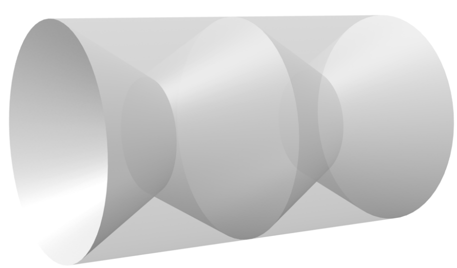
(b)

Figure 1: A body invisible in the horizontal direction constructed using thin flat mirrors.

To obtain invisibility in three dimensions, it is sufficient to ‘stretch’ the body on Fig. 1 (a) in the third direction: observe that the resulting body shown on Fig. 1 (b) is invisible in the horizontal direction.

The body shown on Fig. 1 (b) is not connected, and its use in practice might be limited. We can create connected bodies by gluing four invisible bodies together as shown on Fig. 2 (a): four bodies invisible in the vertical direction (shown in different colour) are attached to each other to form a connected construction. Another way to obtain a connected body is to rotate the two-dimensional body on Fig. 1 (a) around its horizontal axis of symmetry. We then obtain the body on Fig. 2 (b) which is connected and is invisible in the horizontal direction, however, it is not polyhedral.

(a)



(b)

Figure 2: Connected bodies invisible in one direction

We have seen that it is possible to construct bodies invisible in one direction by using simple polyhedral constructions. In the next section we will see that it is possible to construct a body invisible in two directions. We have to, however, resort to more complicated shapes, namely, parabolas. It is yet unknown whether polyhedral bodies invisible in two or more directions exist.

Parabolic constructions and invisibility in two directions

A parabolic body on Fig. 3 looks very much like the one on Fig. 1, however, differs with it in a subtle but important way. The former construction is based on two pairs of co-focal parabolas which all have a common axis. That way, a particle moving in a horizontal direction (along parabolas' axis) after hitting the first parabolic segment is redirected towards the first focus; it then bounces off the next segment symmetrically preserving the original direction of movement, and, finally, bounces two more times from the next two segments, being symmetrically redirected to its original trajectory.

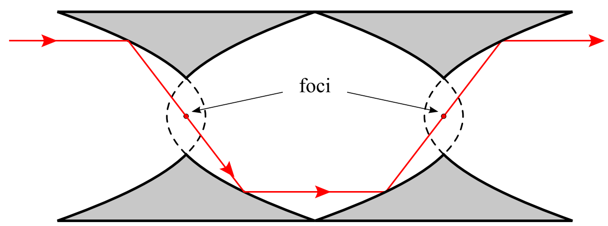


Figure 3: Parabolic construction invisible in one direction.

The two-dimensional body on Fig. 3 is hence invisible in the horizontal direction. We can use the same trick as in the previous section and 'stretch' this body in the third direction to obtain a three-dimensional body invisible in one direction. On Fig. 4 (a) one segment of the body from Fig. 3 is stretched into the third direction, and on Fig. 4 (b) we see the same segment rotated by 90°. On Fig. 4 (c) we see an intersection of such segments. If we now assemble four such bodies as shown on Fig. 5, the resulting construction is invisible in two orthogonal directions (the respective surfaces that work in each of these directions are shown in blue and red colours). We refer the reader to (Plakhov & Roshchina, 2011a) for more details and a proof of invisibility for such a body.

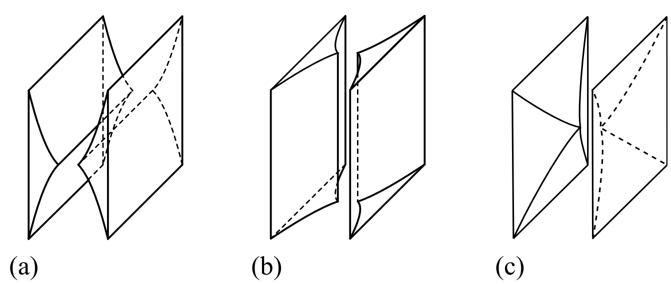


Figure 4: A body invisible in the horizontal direction constructed using thin flat mirrors.

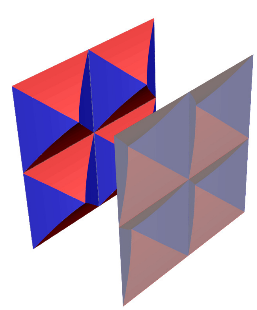


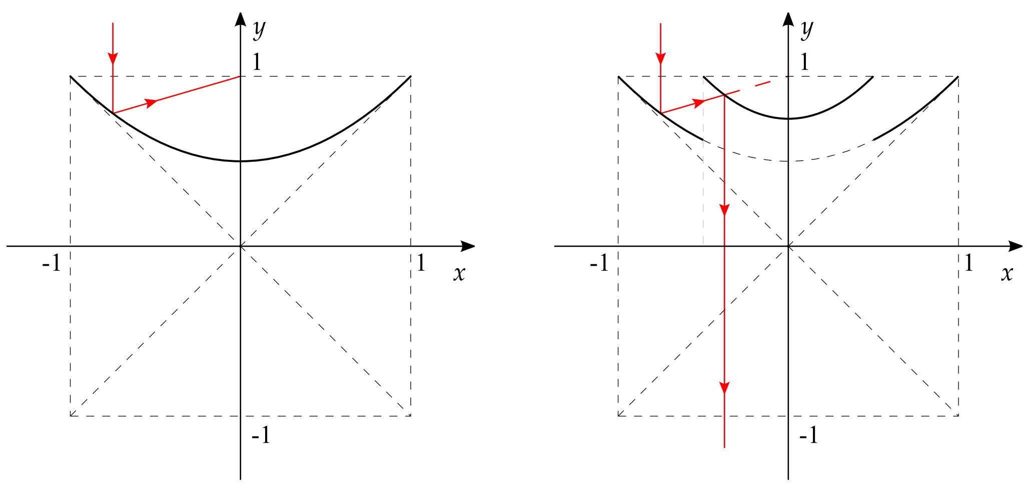
Figure 5: A body invisible in two directions.

What's about a higher number of directions of invisibility? Is is possible to construct a body invisible in two directions in two-dimensional case and in three directions in three-dimensional case? We partially answer this question in the next section, constructing fractal bodies invisible in two and three directions in the relevant spaces.

A thin fractal body invisible in two orthogonal directions

We start with a two-dimensional body invisible in two orthogonal directions. For the clarity of exposition, we assume that the directions of invisibility are parallel to the x- and y-axes.

We construct our body inside the unit square centered at zero. Consider a thin parabolic 'mirror' shown on Fig. 6 (a): it stretches between the two upper vertices of the square and has got focus at (0,1). A particle moving downwards would be reflected towards the focal point after hitting the mirror due to the reflective property of a parabola.



(a) (b)

Figure 6: Fractal body invisible in two directions: (a) constructing the first parabolic mirror; (b) adding a similar confocal mirror.

We add one more parabolic mirror to our construction on Fig. 6 (b). This mirror is similar to the previous one, only two times smaller, while the foci of the corresponding parabolas coincide at $(0, 1)$. Now all particles that go in the downward direction and pass the line segment $[-1, -\frac{1}{2}] \times \{1\})$ (as well as $[\frac{1}{2}, 1] \times \{1\})$), are first reflected from the bigger parabola towards the focal point, move towards the smaller one, and after the second collision are redirected downwards. If we remove a piece of the bigger parabola that is directly beneath the smaller mirror, the resulting body is not obstructing the further movement of the particles, and they leave the body with the same velocity as they had before entering the body (see Fig. 6 (b)).

If we repeat this construction process infinitely, adding figures similar to p_1 and cutting the middle sections of the relevant parabolas out, we obtain a sequence of parabolic mirror segments. This mirror sequence is plotted in Fig. 7 (a). If we now add a symmetric sequence to the bottom of the square, the resulting construction is a fractal body invisible in the vertical (see Fig. 7 (b)).

Indeed, since the lower part of our body is symmetric to the upper part, it is redirecting the particles back to their original trajectories.

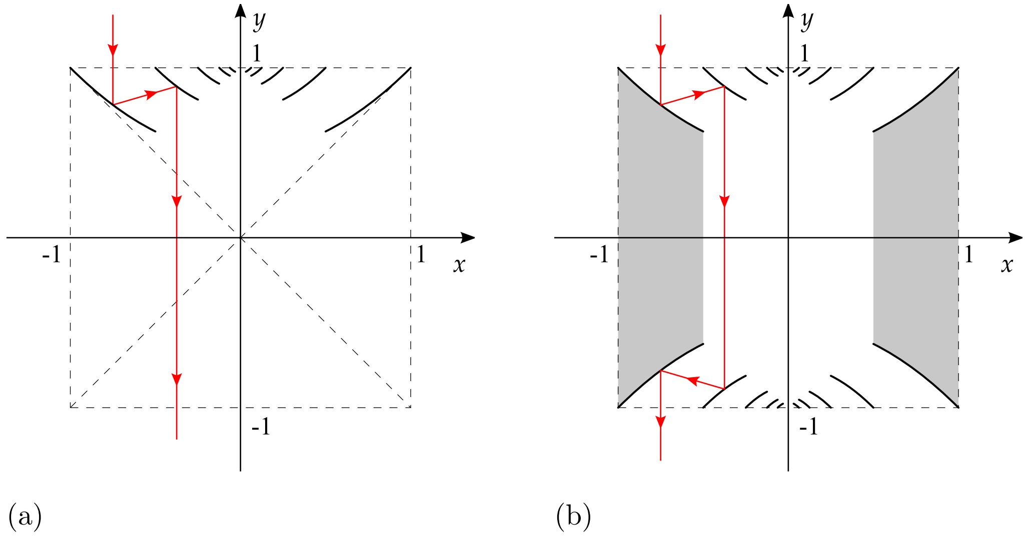


Figure 7: Fractal body invisible in two directions: (a) the basic fractal construction; (b) body invisible in the vertical direction.

Observe that the area greyed in Fig. 7 (b) is completely 'shaded' from the particles moving parallel to the y-axis. We can hence use this area to make our body invisible in a second direction. We simply add one more construction identical to the original one, but rotated by 90°. It is not difficult to observe that it does not overlap the original construction, and makes the body invisible in the horizontal direction. Also observe that the four grey blocks in the corners of the square (see Fig. 8), are never accessed by the particles moving in the directions of invisibility. We can hence include these areas into our body.

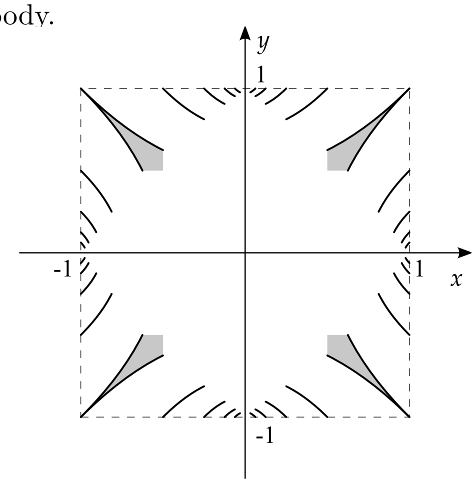


Figure 8: A thin fractal body invisible in two orthogonal directions

We have therefore constructed a body invisible in two orthogonal directions in two-dimensional space. It is possible to generalize this construction and obtain bodies invisible in any two given directions: an interested reader is invited to check (Plakhov & Roshchina, 2011a) for more details. It is also shown in (Plakhov & Roshchina, 2011a) that it is possible to replace the infinitely thin mirrors with segments of nonzero thickness.

A body invisible in 3 directions

Using the two-dimensional construction described in the previous section, we can now describe a three-dimensional body invisible in 3 orthogonal directions.

We are going to define three bodies B_x, B_y, B_z, invisible in the directions x, y, z, respectively, and then take their union.

First we take the two-dimensional body $\mathcal{A}_{yz}$ in the yz-plane, as described in the previous section. It corresponds to 2 *orthogonal* directions (parallel to the y- and z-axes) and is inscribed in the square $[-1, 1]^2$. We take the direct product

$$\tilde{\mathcal{A}}_{yz} = \left(\left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right] \right) \times \mathcal{A}_{yz};$$

the resulting three-dimensional body $\tilde{\mathcal{A}}_{yz}$ is also invisible in the z-direction.

Let $B_{yz} = \tilde{\mathcal{A}}_{yz} \cap \Pi_z$ (Fig. 9 (b)), where $\Pi_z = \{(x, y, z) : \ |z| \geq |x|, \ |z| \geq |y|\}$ is the union of two pyramids with the bases at the top and the bottom of the cube respectively, and the vertices at zero. Then we analogously define the body B_{xz} and take

$$B_z = B_{yz} \cup B_{xz}.$$

The bodies B_x and B_y are defined in a similar way, and finally,

$$B = B_x \cup B_y \cup B_z.$$

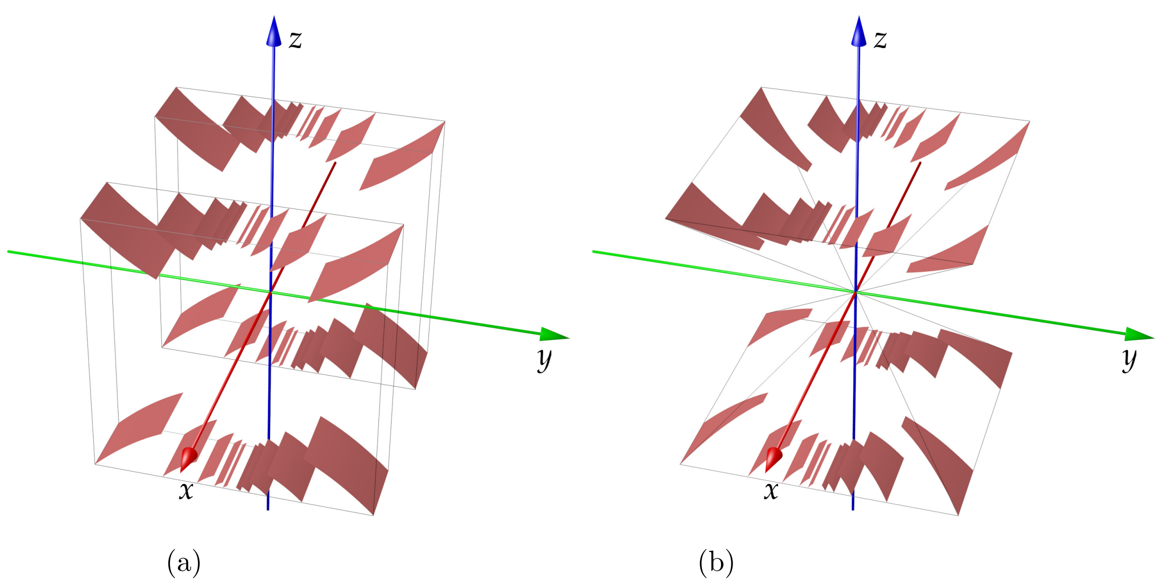
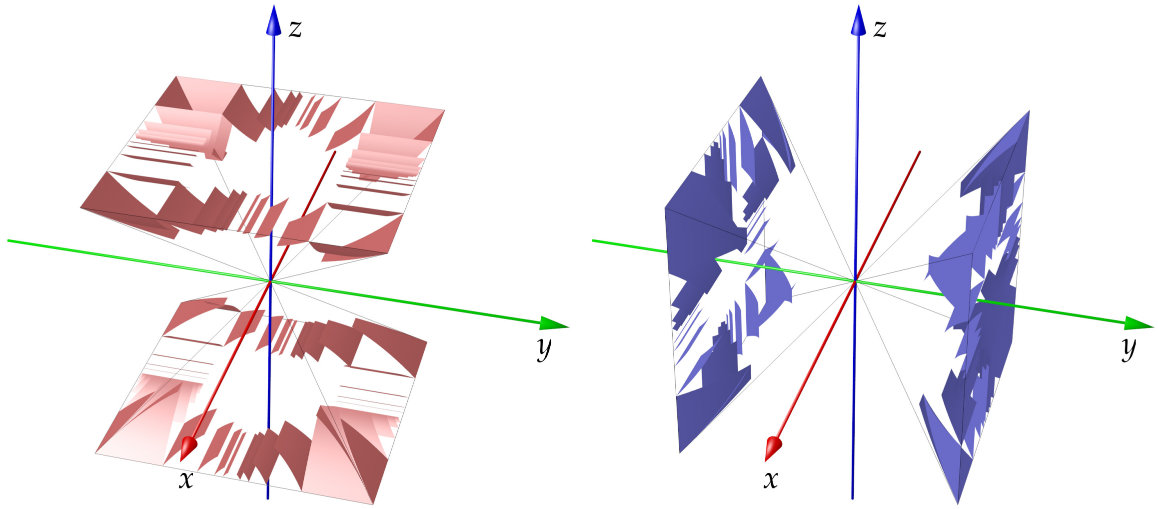


Figure 9: The bodies (a) $\tilde{\mathcal{A}}_{yz}$ and (b) B_{yz}.

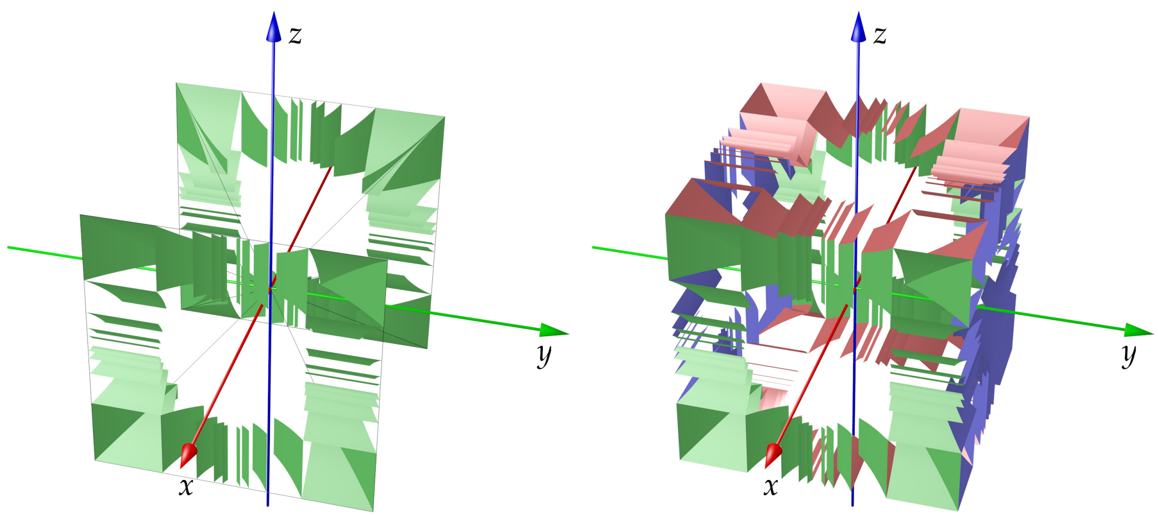
The bodies B_z, B_y and B_x are shown on Fig. 10 (a), Fig. 10 (b) and Fig. 11 (a) respectively. The body B is shown on Fig. 11 (b).

We do not know if it is possible to generalize our construction to 3 non-orthogonal directions.



(a) (b)

Figure 10: Non-overlapping bodies invisible in different directions: (a) along the z-axis; (b) along the y-axis



(a) (b)

Figure 11: A body invisible in the direction along the x-axis (a) and a body invisible in 3 directions (b)

Summary

While bodies invisible in several directions were successfully constructed, several questions related to invisibility remain unanswered. Can we construct a body invisible in n directions in n-dimensional space without using any fractal constructions? Are there bodies invisible in more than n directions in n-dimensional space? What is the maximal number of directions of invisibility? How to introduce an adequate 'measure of invisibility' for a body observed in all directions and find the 'most invisible' body?

There is an intriguing observation related to the existing constructions. There exist connected (and even homeomorphic to the ball) bodies invisible in 1 direction (Aleksenko & Plakhov, 2009). The body invisible in 2 directions found in (Plakhov & Roshchina, 2011a) is disconnected. The body invisible in 3 directions has an infinite number of connected components. We wonder if the increased complexity of the shape is the cost one should pay for the increased number of directions, or whether it is just an artifact of the particular constructions.

These problems are easy to understand, and the existing results can be explained by using only basic school math. However, there are no *tools* or *techniques* for constructing invisible bodies, and this makes the subject even more exciting.

Acknowledgements

This work was partly supported by the Center for Research and Development in Mathematics and Applications (CIDMA) from the "*Fundação para a Ciência e a Tecnologia*" (FCT), cofinanced by the European Community Fund FEDER/POCTI, and by the FCT research project PTDC/MAT/113470/2009.

References

Aleksenko, A., & Plakhov, A. (2009). *Bodies of zero resistance and bodies invisible in one direction.* Nonlinearity **22**, 1247-1258.

Bucur, D., & Buttazzo, G. (2005). *Variational Methods in Shape Optimization Problems.* Birkhäuser.

Buttazzo, G., & Kawohl, B. (1993). *On Newton's problem of minimal resistance.* Math. Intell. **15**, 7-12.

Chang, K. (2007). *Light Fantastic: Flirting With Invisibility.* The New York Times, June 12.

Comte, M., & Lachand-Robert, T. (2001). *Newton's problem of the body of minimal resistance under a single-impact assumption.* Calc. Var. Partial Differ. Equ. **12**, 173-211.

Ergin, T., Stenger, N., Brenner, P., Pendry, J. B., & Wegener, M. (2010). *Three-Dimensional Invisibility Cloak at Optical Wavelengths.* Science **328**, 337–339.

Inami, M., Kawakami, N., & Tachi, S. (2003). *Optical Camouflage Using Retro-Reflective Projection Technology* Proceedings of the 2nd IEEE/ACM International Symposium on Mixed and Augmented Reality. p. 348.

Lachand-Robert, T., & Oudet, E. (2006). *Minimizing within convex bodies using a convex hull method.* SIAM J. Optim. **16**, 368–379.

Lachand-Robert, T., & Peletier, M. A. (2001). *Newton's problem of the body of minimal resistance* *in the class of convex developable functions*. Math. Nachr. **226**, 153-176.

Newton, I. (1687). *Philosophiae naturalis principia mathematica*. 1687.

Plakhov, A. (2009). *Scattering in billiards and problems of Newtonian aerodynamics*. Russ. Math. Surv. **64**, 873-938.

Plakhov, A., & Aleksenko, A. (2010). *The problem of the body of revolution of minimal resistance*. ESAIM Control Optim. Calc. Var. **16**, 206-220.

Plakhov, A., & Roshchina, V. (2011a). *Invisibility in billiards*. Nonlinearity **24**, 847–854.

Plakhov, A., & Roshchina, V. (2011b). *Fractal bodies invisible in 2 and 3 directions*, arXiv:1107.5667.

Schurig, D., Mock, J. J., Justice, B. J., Cummer, S. A., Pendry, J. B., Starr, A. F., & Smith, D. R. (2006). *Metamaterial Electromagnetic Cloak at Microwave Frequencies*, Science **314**, 977.

Valentine, J., Zhang, S., Zentgraf, T., Ulin-Avila, E., Genov, D. A., Bartal, G., & Zhang, X. (2008). *Three-dimensional optical metamaterial with a negative refractive index*. Nature, **455**.

RECREATIONAL MATHEMATICS COLLOQUIUM II

April 27th - April 30th 2011

University of Évora

Invited Speakers

Colin Wright, UK
David Singmaster, UK
Keith Devlin, USA
Lennart Green, Sweden
Richard Nowakowski, Canada
Robin Wilson, UK

Scientific Committee

David Singmaster, UK
João Pedro Neto, Portugal
Jorge Buescu, Portugal
Jorge Nuno Silva, Portugal
Keith Devlin, USA
Nuno Crato, Portugal
Richard Nowakowski, Canada
Robin Wilson, UK
Sandra Vinagre, Portugal

Organizing Committee

Alda Carvalho, Portugal
Ana Santos, Portugal
Carlos P. Santos, Portugal
Jorge Nuno Silva, Portugal
Liliana Monteiro, Portugal
Sandra Vinagre, Portugal

Contactos

http://ludicum.org/rm11/
rm11@ludicum.org

Este Colóquio está em processo de acreditação pelo Conselho Científico e Pedagógico da Formação Contínua nos grupos de docência 230, do 2° ciclo do Ensino Básico, e 500, do 3° ciclo do Ensino Básico e Secundário.

Organizers

Universidade de Évora — Ludus — Museu de Ciência, Universidade de Lisboa — Ciência Viva, Agência Nacional para a Cultura Científica e Tecnológica

Sponsors

cima-UE, Centro de Investigação em Matemática e Aplicações da Universidade de Évora — CMAF UL — CIUHCT — spm, Sociedade Portuguesa de Matemática — Cemapre — Associação de Professores de Matemática